Dottorato di Ricerca in Ingegneria dei Sistemi
Corso: Modellistica e Controllo di Robot con Giunti Flessibili
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Part 1: Modeling and Control of Robots with Elastic Joints

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Outline

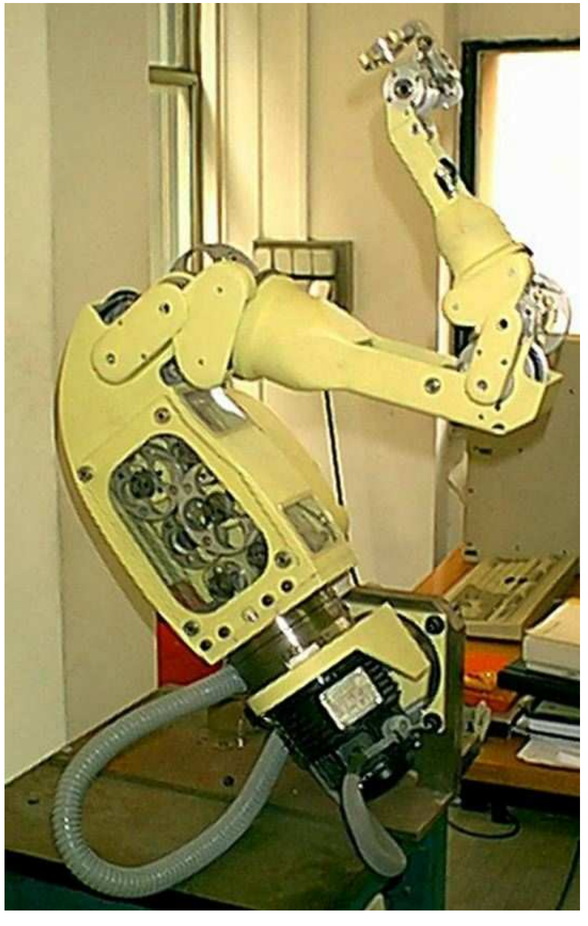
- Motivation for considering joint flexibility/elasticity
- Dynamic model of robots with joints of constant stiffness (= elastic joints (EJ))
  - reduced, singularly perturbed, or complete model
- Inverse dynamics
- Sensing requirements and formulation of control problems
- Controllers for regulation tasks
  - motor PD + constant or on-line gravity compensation
- Controllers for trajectory tracking tasks
  - feedback linearization and two-time scale designs
- Some modeling and control extensions
  - mixed rigid/elastic case
  - dynamic feedback linearization of the complete model
- Research issues
- References

Motivation

- in industrial robots, the presence of transmission elements such as
  - harmonic drives and transmission belts (typically, Scara arms)
  - long shafts (e.g., last 3-dofs of Puma)

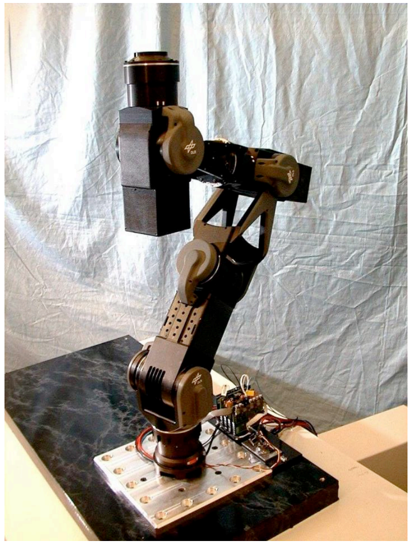
  introduce flexibility effects between actuating inputs and driven outputs
- desire of mechanical compliance in arms (or in legs for locomotion) leads to the use of elastic transmissions in robots for safe physical interaction with humans
  - actuator relocation plus cables and pulleys
  - harmonic drives and lightweight (but rigid) link design
  - redundant (macro-mini or parallel) actuation
  - variable elasticity/stiffness actuation (VSA)
- these phenomena are captured by modeling the **flexibility at the robot joints**
- neglected joint flexibility limits dynamic performance of controllers (vibrations, poor tracking, chattering during environment contact)

Robots with joint elasticity — DEXTER



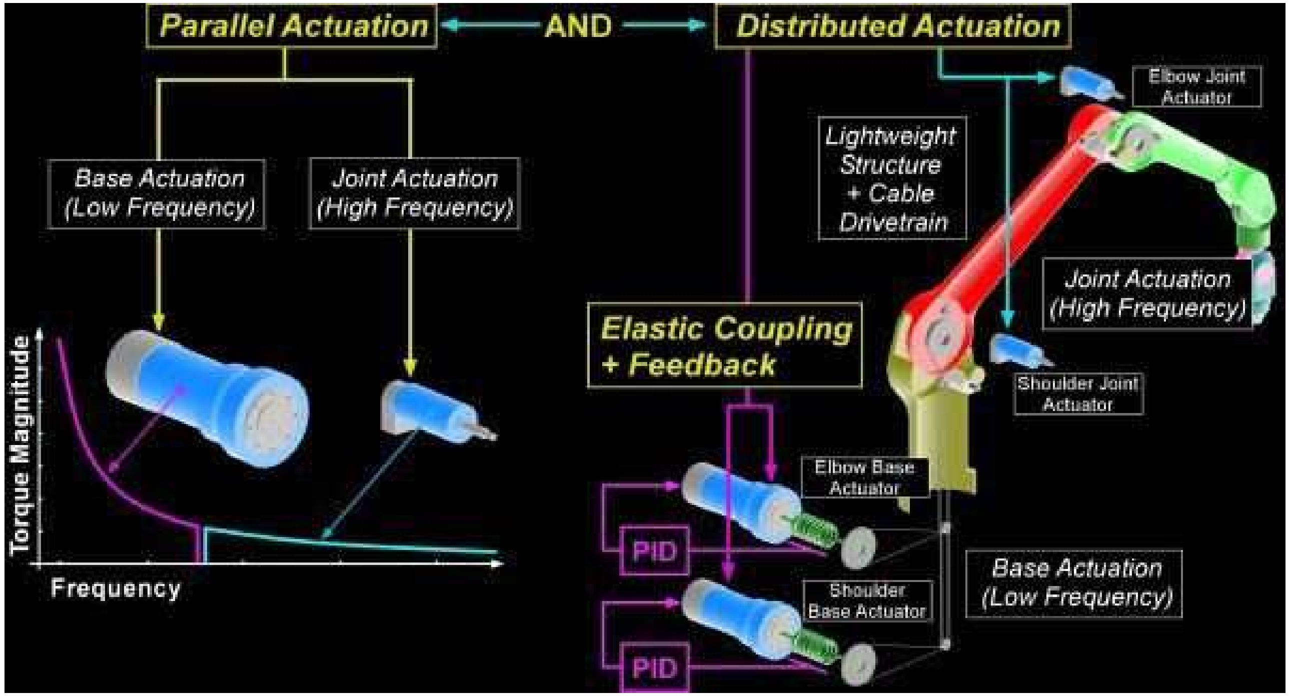
- 8R-arm by Scienzia Machinale
- DC motors with reductions for joints 1,2
- DC motors with reductions, steel cables and pulleys for joints 3–8 (all located in link 2)
- encoders on motor sides

Robots with joint elasticity — DLR and KUKA LWR



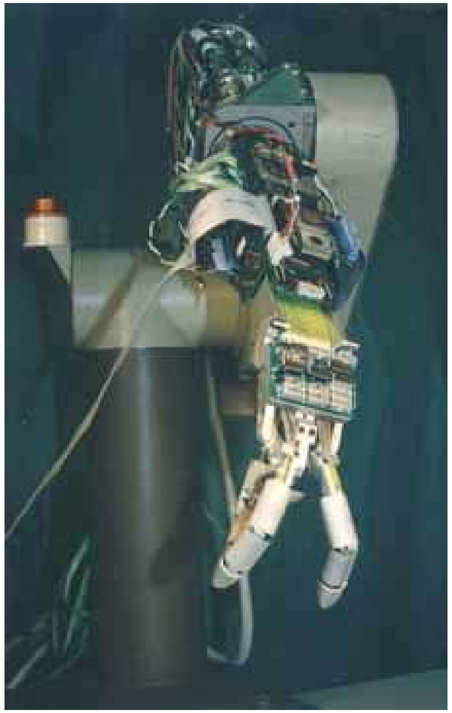
- LWR-II and LWR-III by DLR Institute of Robotics and Mechatronics, and the latest industrial version by KUKA
- 7R robot arms with DC brushless motors and harmonic drives
- encoders on motor and link sides, joint torque sensors
- modular, lightweight ($<$ 14 kg), with 7 kg payload!

Robots with joint elasticity — DECMMA



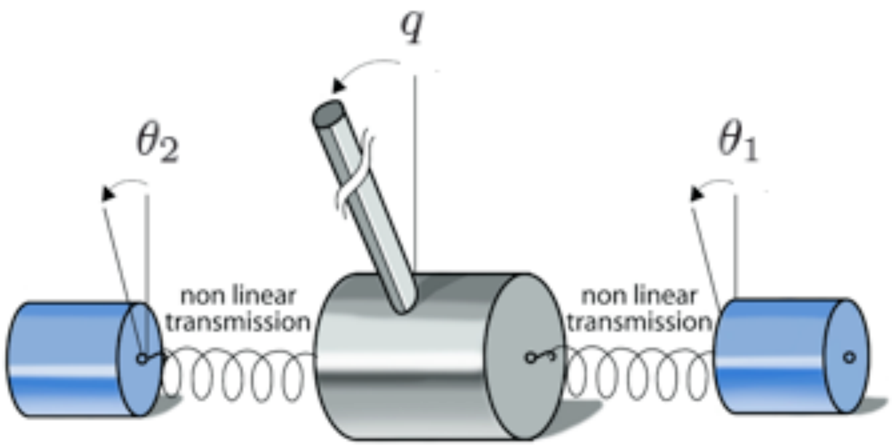
- 2R and 4R prototype arms by Stanford University Robotics Laboratory
- parallel macro (at base, with elastic coupling) – mini (at joints) actuation

Robots with joint elasticity — UB Hand



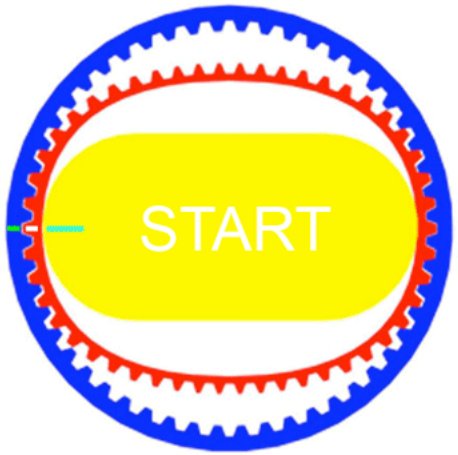
- dextrous hand mounted as end-effector of a Puma robot
- tendon-driven (static compliance in the grasp)

Robots with Variable Stiffness actuation — VSA-II



- 1-dof prototype by University of Pisa (being extented to 3R robot arm)
- two DC motors, with nonlinear and variable stiffness transmission
- linear springs, with nonlinear geometric four-bar linkages

Joint elasticity in harmonic drives — industrial robots

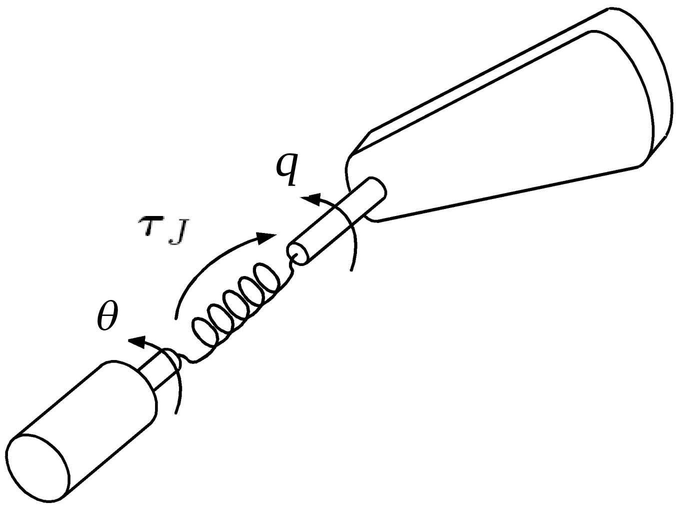


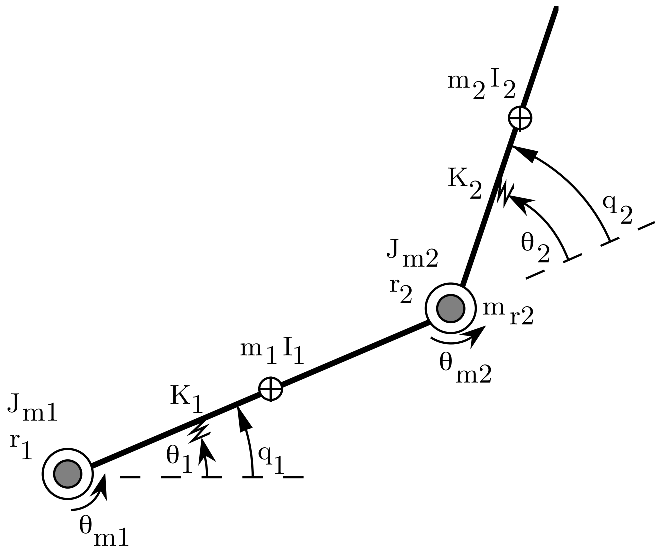
- compact, in-line, high reduction (1:200), power efficient transmission element
- teflon teeth of flexspline introduce small angular displacement

video HD

Dynamic modeling

- open-chain robot with N (rotary or prismatic) elastic joints and N rigid links, driven by electrical actuators
- Lagrangian formulation using motor variables $\theta \in I\!R^N$ (as reflected through reduction ratios) and link variables $q \in I\!R^N$ as generalized coordinates





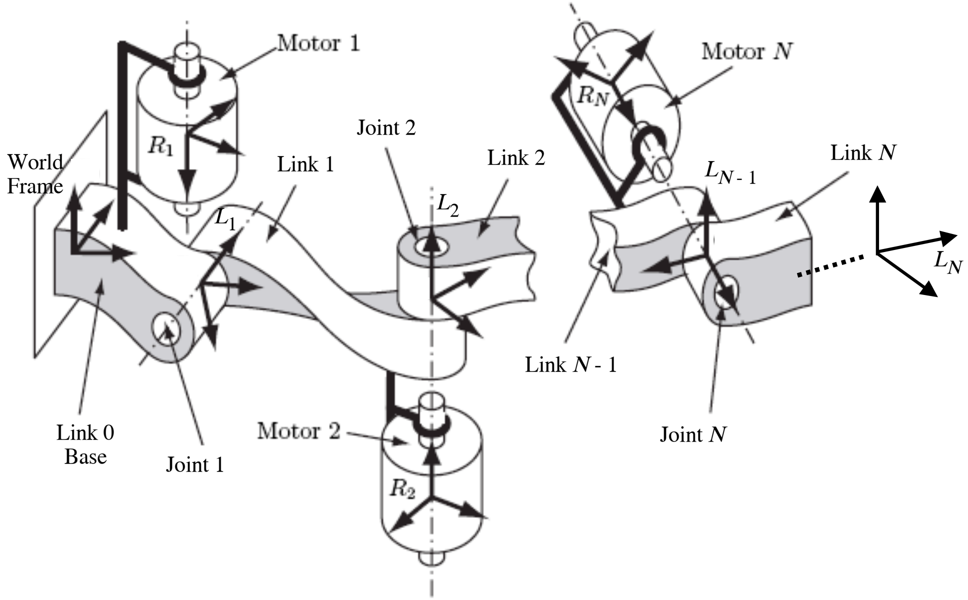
- standing assumptions

A1) small displacements at joints (linear elasticity domain)

A2) axis-balanced motors (i.e., center of mass of rotors on rotation axes)

- further assumption on location of actuators in the kinematic chain

A3) each motor is mounted on the robot in a position preceding the driven link



- link (linear + angular) kinetic energy

$$T_\ell = \sum_{i=1}^{N} T_{\ell_i} = \sum_{i=1}^{N} \left(T_{L,\ell_i} + T_{A,\ell_i}\right) = \sum_{i=1}^{N} \frac{1}{2}\left(m_{\ell_i} v_{c,\ell_i}^T v_{c,\ell_i} + \omega_{\ell_i}^T I_{\ell_i} \omega_{\ell_i}\right)$$
$$= \frac{1}{2}\dot{q}^T M_\ell(q)\dot{q}$$

- motor linear kinetic energy —the mass $m_{m_i} = m_{s_i} + m_{r_i}$ of each motor (stator + rotor) is just an additional mass of the carrying link

$$T_{L,m} = \sum_{i=1}^{N} T_{L,m_i} = \sum_{i=1}^{N} \frac{1}{2} m_{m_i} v_{c,m_i}^T v_{c,m_i} = \frac{1}{2}\dot{q}^T M_m(q)\dot{q}$$

- summing up, a symmetric inertia matrix $M(q) > 0$ results

$$T_\ell + T_{L,m} = \frac{1}{2}\dot{q}^T \left(M_\ell(q) + M_m(q)\right)\dot{q} = \frac{1}{2}\dot{q}^T M(q)\dot{q}$$

- link and motor potential energy due to gravity

$$U_g = U_{g,\ell} + U_{g,m} = \sum_{i=1}^{N} \Big(U_{g,\ell_i}(q) + U_{g,m_i}(q) \Big) = U_g(q)$$

- A2) $\Rightarrow$ both M, U_g are independent from θ
- potential energy due to joint elasticity

$$U_e = \frac{1}{2}(q-\theta)^T K (q-\theta)$$

with diagonal, positive definite $N \times N$ matrix K of joint stiffness

- simplifying assumption $\Rightarrow$ reduced dynamic model of (Spong, 1987)

A4) angular kinetic energy of each motor is due only to its own spinning

$$T_{A,m} = \frac{1}{2}\dot{\theta}^T B\,\dot{\theta}$$

with constant, diagonal, positive definite motor inertia matrix B (reflected through the reduction ratios: $B_i = J_{mi} r_i^2$, $i = 1, \ldots, N$)

- system Lagrangian

$$\begin{aligned} L &= T - U = (T_\ell + T_{L,m} + T_{A,m}) - (U_g + U_e) \\ &= \tfrac{1}{2}\dot{q}^T M(q)\dot{q} + \tfrac{1}{2}\dot{\theta}^T B\,\dot{\theta} - U_g(q) - \tfrac{1}{2}(q-\theta)^T K(q-\theta) \\ &= L(q, \theta, \dot{q}, \dot{\theta}) \end{aligned}$$

Euler-Lagrange equations

- given the set of generalized coordinates $p = (q^T \; \theta^T)^T \in \mathbb{R}^{2N}$, the Lagrangian L satisfies the usual vector equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}}\right)^T - \left(\frac{\partial L}{\partial p}\right)^T = u$$

  being $u \in \mathbb{R}^{2N}$ the non-conservative forces/torques performing work on p

- assuming no dissipative terms and no external forces (acting on links), since the motor torques $\tau \in \mathbb{R}^N$ only perform work on the motor variables θ we obtain

$$\begin{aligned} M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) + K(q-\theta) &= 0 \qquad \text{link equation} \\ B\ddot{\theta} + K(\theta - q) &= \tau \qquad \text{motor equation} \end{aligned}$$

  with centrifugal/Coriolis terms $C(q,\dot{q})\dot{q}$ and gravity terms $g(q) = (\partial U_g/\partial q)^T$

Coriolis/centrifugal terms

- being the generalized coordinates $p = (q^T \ \theta^T)^T$, these quadratic terms in the generalized velocity $\dot{p}$ are computed by (symbolic) differentiation of the elements of the $2n \times 2N$ robot inertia matrix

$$\mathcal{M}(p) = \begin{pmatrix} M(q) & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} \mathcal{M}_1 \ \mathcal{M}_2 & \vdots & \mathcal{M}_{2N} \end{pmatrix} \qquad \text{(columns)}$$

  using the Christoffel symbols (of the second type):

$$\mathcal{C}(p,\dot{p})\dot{p} = \text{col}\left\{\dot{p}^T \mathcal{C}_i(p)\dot{p}\right\}$$

$$\mathcal{C}_i(p) = \frac{1}{2}\left(\left(\frac{\partial \mathcal{M}_i}{\partial p}\right) + \left(\frac{\partial \mathcal{M}_i}{\partial p}\right)^T - \left(\frac{\partial \mathcal{M}}{\partial p_i}\right)\right) \qquad i = 1, 2, \ldots, 2N$$

- thanks to the simple structure of $\mathcal{M} = \mathcal{M}(q)$, the computation is relevant only for the upper left block $M(q) \Rightarrow$ only $C(q,\dot{q})\dot{q}$ in link equation

Model properties

- $\dot{M}(q) - 2C(q,\dot{q})$ is skew-symmetric
- nonlinear dynamic model, but linear in a set of coefficients $a = (a_r, a_K, a_B)$ (including K and B)

$$\begin{aligned} Y_r(q,\dot{q},\ddot{q})\, a_r + \text{diag}\{q-\theta\}\, a_K &= 0 \\ \text{diag}\{\ddot{\theta}\}\, a_B + \text{diag}\{\theta - q\}\, a_K &= \tau \end{aligned}$$

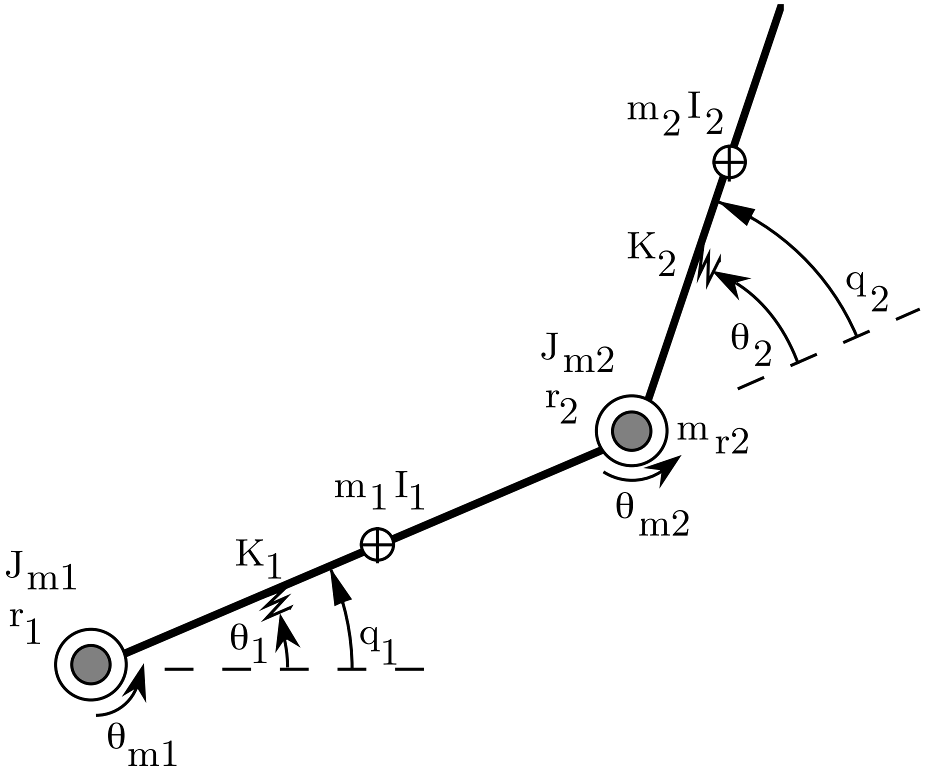
- for $K \to \infty$ (rigid joints): $\theta \to q$ and $K(q-\theta) \to$ finite value, so that the equivalent rigid model is recovered (summing up link and motor equation)

$$\big(M(q) + B\big)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

- there exists a bound on the norm of the gravity gradient matrix

$$\left\| \frac{\partial g(q)}{\partial q} \right\| \le \alpha \quad \Rightarrow \quad \|g(q_1) - g(q_2)\| \le \alpha \|q_1 - q_2\|, \; \forall q_1, q_2 \in I\!R^N$$

. . . work out the dynamic model for a case study



planar 2R arm with elastic joints (without or with gravity)

Singularly perturbed dynamic model

- if joint stiffnesses $K = \text{diag}\{K_1, \dots, K_N\}$ are very large ($\approx$ rigid joints), the system exhibits a two-time scale dynamic behavior in terms of link position (q) and joint deformation torque (z) that can be used for simpler control design
- to show this, we use a linear change of coordinates

$$\begin{pmatrix} q \\ z \end{pmatrix} = \begin{pmatrix} q \\ K(\theta - q) \end{pmatrix}$$

and rewrite the motor acceleration and the second time derivative of the joint deformation torque as

$$\begin{aligned} \ddot{\theta} &= B^{-1}(\tau - z) \\ \ddot{z} &= K(\ddot{\theta} - \ddot{q}) = K\left(B^{-1}(\tau - z) + M^{-1}(q)\left(C(q,\dot{q})\dot{q} + g(q) - z\right)\right) \\ &= KB^{-1}\tau + KM^{-1}(q)\left(C(q,\dot{q})\dot{q} + g(q)\right) - K\left(B^{-1} + M^{-1}(q)\right)z \end{aligned}$$

- from K, we can extract a common large scalar factor $\dfrac{1}{\epsilon^2} \gg 1$ so that

$$K = \frac{1}{\epsilon^2}\hat{K} = \frac{1}{\epsilon^2}\,\text{diag}\{\hat{K}_1, \ldots, \hat{K}_N\}$$

with $\hat{K}_i$ of similar (moderate) amplitude

- the second dynamic equation becomes

$$\epsilon^2 \ddot{z} = \hat{K}B^{-1}\tau + \hat{K}M^{-1}(q)\left(C(q,\dot{q})\dot{q} + g(q)\right) - \hat{K}\left(B^{-1} + M^{-1}(q)\right)z \quad (*)$$

and represents the fast dynamics associated with the elastic joints, while

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = z$$

represents the slow dynamics of the links

- time scaling is made explicit by introducing the fast time variable $\sigma = \dfrac{t}{\epsilon}$ in $(*)$

$$\epsilon^2 \ddot{z} = \epsilon^2 \frac{d^2 z}{dt^2} = \frac{d^2 z}{d\sigma^2}$$

Inverse dynamics

- given a sufficiently smooth link trajectory $q_d(t)$, together with a number of its time derivatives, compute the required motion torque $\tau_d(t)$
- the associated motor trajectory $\theta_d(t)$ is needed
- the motor position is computed from the link equation as

$$\theta_d = q_d + K^{-1}\left(M(q_d)\ddot{q}_d + C(q_d, \dot{q}_d)\dot{q}_d + g(q_d)\right)$$

- motor velocity is computed from the first time derivative of link equation

$$\dot{\theta}_d = \dot{q}_d + K^{-1}\Big(M(q_d)q_d^{[3]} + \dot{M}(q_d)\ddot{q}_d + \dot{C}(q_d, \dot{q}_d)\dot{q}_d + C(q_d, \dot{q}_d)\ddot{q}_d + \dot{g}(q_d)\Big)$$

using the notation $x^{[i]} = \frac{d^i x}{dt^i}$

- motor acceleration is computed from the second time derivative of link equation

$$\ddot{\theta}_d = \ddot{q}_d + K^{-1}\Big(M(q_d)q_d^{[4]} + 2\dot{M}(q_d)q_d^{[3]} + \ddot{M}(q_d)\ddot{q}_d + \ddot{C}(q_d,\dot{q}_d)\dot{q}_d + 2\dot{C}(q_d,\dot{q}_d)\ddot{q}_d + C(q_d,\dot{q}_d)q_d^{[3]} + \ddot{g}(q_d)\Big)$$

- finally, the needed torque is computed from the motor equation by substitution

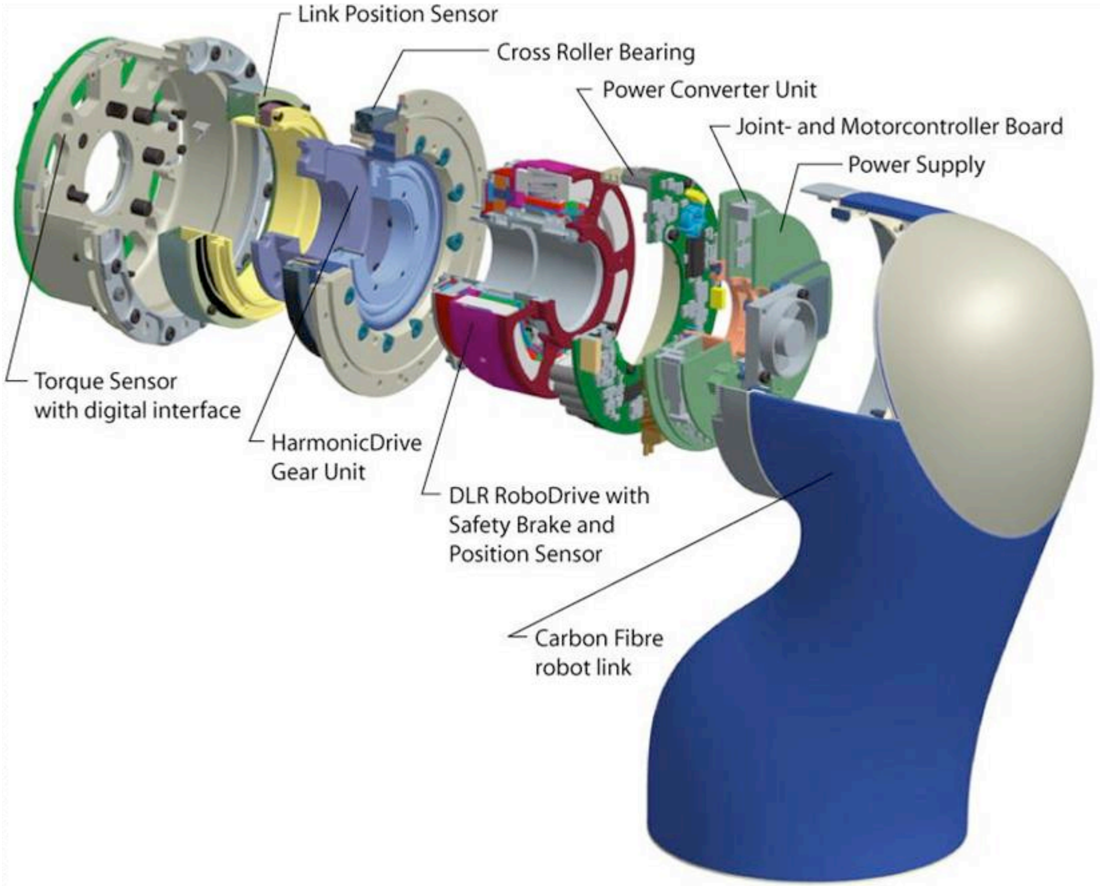
$$\tau_d = B\ddot{\theta}_d + K(\theta_d - q_d) = BK^{-1}\Big(M(q_d)q_d^{[4]} + \ldots + \ddot{g}(q_d)\Big) + (M(q_d) + B)\,\ddot{q}_d + C(q_d,\dot{q}_d)\dot{q}_d + g(q_d)$$

- this Lagrangian-based scheme may become computationally heavy for large N
- a recursive $O(N)$ Newton-Euler inverse dynamics algorithm may be set up, by including in the forward recursions also the linear/angular link jerks (third derivatives) and snaps (fourth derivatives) and in the backward recursions also the first and second derivatives of the link forces/torques

Sensing requirements

- full state feedback requires sensing of four variables: q, θ (link/motor position) and $\dot{q}$, $\dot{\theta}$ (link/motor velocity) $\Rightarrow$ $4N$ state variables for a N-dof EJ robot
- only motor variables $(\theta, \dot{\theta})$ are available with standard sensing arrangements (encoder $+$ tachometer on the motor axis)
- velocities also through numerical differentiation of high-resolution encoders
- advanced systems have also measures beyond the elasticity of the joints (e.g., link position q and joint torque $\tau_J = K(q-\theta)(= -z)$ sensors in DLR LWRs)

Exploded view of a DLR LWR-III joint



Control problems

- regulation to a constant equilibrium configuration $(q, \theta, \dot{q}, \dot{\theta}) = (q_d, \theta_d, 0, 0)$
  - only the desired link position q_d is assigned, while θ_d has to be determined
  - q_d may come from the kinematic inversion of a desired cartesian pose x_d
  - direct kinematics of EJ robots is a function of link variables: $x = \text{kin}(q)$
- tracking of a sufficiently smooth trajectory $q = q_d(t)$
  - the associated motor trajectory $\theta_d(t)$ has to be determined
  - again, $q_d(t)$ is uniquely associated to a desired cartesian trajectory $x_d(t)$
- other relevant planning/control problems not considered here include
  - rest-to-rest trajectory planning in given time T
  - Cartesian control (regulation or tracking directly defined in the task space)
  - force/impedance/hybrid control of EJ robots in contact with the environment

Regulation

— a simple linear example

- two elastically coupled masses (motor/link) driven on one side (Quanser LEJ)

- dynamic model (without damping/friction effects)

$$m\ddot{q} + k(q - \theta) = 0 \qquad b\ddot{\theta} + k(\theta - q) = \tau$$

- using Laplace transform, we can define two input-output transfer functions of interest from the force input τ to ...
  - the position θ of the first mass (collocated), representing the motor
  - the position q of the second mass (non-collocated), representing the link

Transfer functions of interest

- motor transfer function

$$P_{\text{motor}}(s) = \frac{\theta(s)}{\tau(s)} = \frac{ms^2 + k}{mbs^2 + (m+b)k} \cdot \frac{1}{s^2}$$

  - unstable system with zeros, but passive (zeros always precede poles on the imaginary axis) $\longrightarrow$ stabilization can be achieved via output (θ) feedback

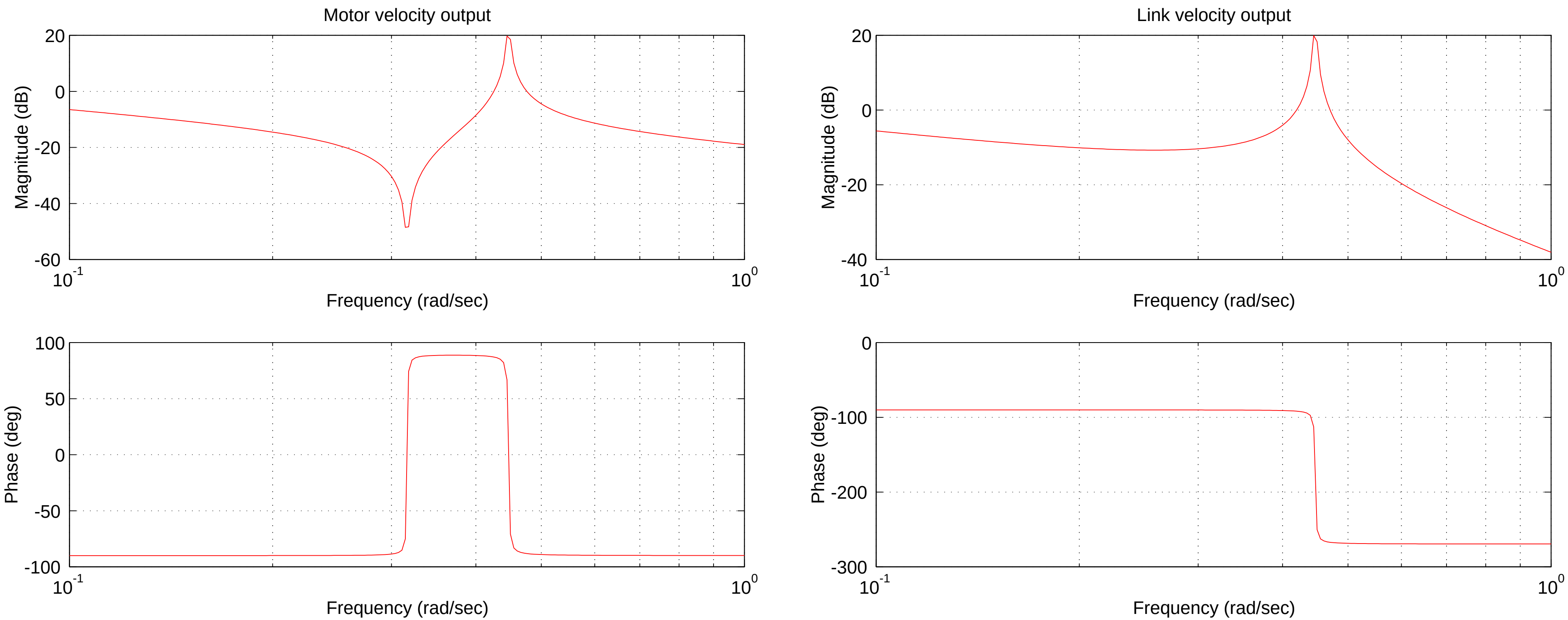
- link transfer function

$$P_{\text{link}}(s) = \frac{q(s)}{\tau(s)} = \frac{k}{mbs^2 + (m+b)k} \cdot \frac{1}{s^2}$$

  - unstable but controllable system as before ($\longrightarrow$ any pole assignment via full state feedback), but now without zeros!

- with damping, pole/zero pairs are moved to the left-hand side of C-plane

Typical frequency response of a single elastic joint



- antiresonance/resonance behavior on motor velocity output, pure resonance on link velocity output (weak or no zeros)

Feedback strategies with reduced measurements

1) $\tau = u_1 - (k_{P\ell} q + k_{D\ell} \dot{q})$ (link PD feedback)

$$W_{\ell\ell}(s) = \frac{q(s)}{u_1(s)} = \frac{k}{mbs^4 + (m+b)ks^2 + kk_{D\ell}s + kk_{P\ell}}$$

always unstable (spring damping/friction leads to small stability intervals)

2) $\tau = u_2 - (k_{Pm} \theta + k_{Dm} \dot{\theta})$ (motor PD feedback)

$$W_{mm}(s) = \frac{k}{mbs^4 + mk_{Dm}s^3 + [m(k + k_{Pm}) + bk]\, s^2 + kk_{Dm}s + kk_{Pm}}$$

asympotically stable for $k_{Pm} > 0$, $k_{Dm} > 0$ (Routh criterion) $\longrightarrow$ as in rigid systems!

3) $\tau = u_3 - (k_{P\ell} q + k_{Dm} \dot{\theta})$ (link position and motor velocity feedback)

$$W_{\ell m}(s) = \frac{k}{mbs^4 + mk_{Dm}s^3 + (m+b)ks^2 + kk_{Dm}s + kk_{P\ell}}$$

asympotically stable for $0 < k_{P\ell} < k$, $k_{Dm} > 0$ (limited proportional gain)

4) with $\tau = u_4 - (k_{Pm}\theta + k_{D\ell}\dot{q})$ (motor position and link velocity feedback) the closed-loop system is always unstable

$$\Downarrow$$

- caution must be used in dealing with different output measures
- generalization to a nonlinear multidimensional setting (under gravity) of the most efficient scheme (motor PD feedback)

video Quanser

Regulation with motor PD + feedforward

- for regulation tasks, consider the control law

$$\tau = K_P(\theta_d - \theta) - K_D\dot{\theta} + g(q_d)$$

with symmetric (diagonal) $K_P > 0$, $K_D > 0$, and the motor reference position

$$\theta_d := q_d + K^{-1}g(q_d)$$

Theorem (Tomei, 1991) If

$$\lambda_{min}(K_E) := \lambda_{min}\left(\begin{bmatrix} K & -K \\ -K & K + K_P \end{bmatrix}\right) > \alpha$$

then the closed-loop equilibrium state $(q_d, \theta_d, 0, 0)$ is globally asymptotically stable

Lyapunov-based proof

- closed-loop equilibria ($\dot{q} = \dot{\theta} = \ddot{q} = \ddot{\theta} = 0$) satisfy

$$\begin{aligned} K(q-\theta) + g(q) &= 0 \\ K(\theta - q) - K_P(\theta_d - \theta) - g(q_d) &= 0 \end{aligned}$$

- adding/subtracting $K(\theta_d - q_d) - g(q_d)$ ($= 0$, by definition of θ_d) yields

$$\begin{aligned} K(q-q_d) - K(\theta - \theta_d) + g(q) - g(q_d) &= 0 \\ -K(q-q_d) + (K_P + K)(\theta - \theta_d) &= 0 \end{aligned}$$

or, in matrix form,

$$K_E \begin{bmatrix} q - q_d \\ \theta - \theta_d \end{bmatrix} = \begin{bmatrix} g(q_d) - g(q) \\ 0 \end{bmatrix}$$

- using the Theorem assumption, for all $(q, \theta) \neq (q_d, \theta_d)$ we have

$$\begin{aligned}
\left\| K_E \begin{bmatrix} q - q_d \\ \theta - \theta_d \end{bmatrix} \right\| &\geq \lambda_{min}(K_E) \left\| \begin{bmatrix} q - q_d \\ \theta - \theta_d \end{bmatrix} \right\| \\
&\boxed{>} \alpha \left\| \begin{bmatrix} q - q_d \\ \theta - \theta_d \end{bmatrix} \right\| \geq \alpha \left\| q - q_d \right\| \\
&\geq \left\| g(q_d) - g(q) \right\| = \left\| \begin{bmatrix} g(q_d) - g(q) \\ 0 \end{bmatrix} \right\|
\end{aligned}$$

and hence (q_d, θ_d) is the unique equilibrium configuration

- define the position-dependent energy function

$$P(q, \theta) = \frac{1}{2}(q - \theta)^T K (q - \theta) + \frac{1}{2}(\theta_d - \theta)^T K_P (\theta_d - \theta) + U_g(q) - \theta^T g(q_d)$$

its gradient $\nabla P(q, \theta) = 0$ only at (q_d, θ_d) (using the same above argument) and $\nabla^2 P(q_d, \theta_d) > 0 \Rightarrow (q_d, \theta_d)$ is an absolute minimum for $P(q, \theta)$

- the following is thus a candidate Lyapunov function

$$V(q,\theta,\dot q,\dot\theta) = \frac{1}{2}\dot q^T M(q)\dot q + \frac{1}{2}\dot\theta^T B\dot\theta + P(q,\theta) - P(q_d,\theta_d) \geq 0$$

- its time derivative, evaluated along the closed-loop system trajectories, is

$$\begin{aligned}
\dot V &= \dot q^T M(q)\ddot q + \tfrac{1}{2}\dot q^T \dot M(q)\dot q + \dot\theta^T B\ddot\theta + \left(\dot q - \dot\theta\right)^T K\,(q-\theta) \\
&\quad - \dot\theta^T K_P\,(\theta_d - \theta) + \dot q^T \left(\tfrac{\partial U_g(q)}{\partial q}\right)^T - \dot\theta^T g(q_d) \\
&= \dot q^T \left(-C(q,\dot q)\dot q - g(q) - K(q-\theta) + \tfrac{1}{2}\dot M(q)\dot q + K(q-\theta) + g(q)\right) \\
&\quad + \dot\theta^T\left(u - K(\theta - q) - K(q-\theta) - K_P(\theta_d-\theta) - g(q_d)\right) \\
&= \dot\theta^T \left(K_P\,(\theta_d-\theta) - K_D\dot\theta + g(q_d) - K_P\,(\theta_d-\theta) - g(q_d)\right) \\
&= -\dot\theta^T K_D\dot\theta \leq 0
\end{aligned}$$

where the skew-symmetry of $\dot M - 2C$ has been used

- since $\dot{V} = 0 \iff \dot{\theta} = 0$, the proof is completed using LaSalle Theorem
- substituting $\dot{\theta}(t) \equiv 0$ in the closed-loop equations yields

$$\begin{aligned} M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) + Kq = K\theta &= \text{constant} & (*) \\ Kq = K\theta - K_P(\theta_d - \theta) - g(q_d) &= \text{constant} & (**) \end{aligned}$$

from $(**)$ it follows that $\dot{q}(t) \equiv 0$, which in turn simplifies $(*)$ into

$$g(q) + K(q - \theta) = 0 \qquad (***)$$

- from the first part of the proof, the unique solution to $(**)$–$(***)$ is $q = q_d$, $\theta = \theta_d$ and thus the largest invariant set contained in the set of states such that $\dot{V} = 0 \Rightarrow$ global asymptotic stability of the set point ■

Remarks on regulation control

- if the (minimum) joint stiffness $\min_{i=1,\dots,N} K_i > \alpha$, the Theorem assumption $\lambda_{min}(K_E) > \alpha$ can always be satisfied by increasing $\lambda_{min}(K_P)$
- since the symmetric matrices K and K_P are assumed diagonal, it is sufficient to analyze the scalar case ($N = 1$)

$$\det(\lambda I - K_E) = \lambda^2 - (2K + K_P)\lambda + K K_P$$

$$\Rightarrow \; \lambda_{min}(K_E) = K + \frac{K_P}{2} - \sqrt{K^2 + \left(\frac{K_P}{2}\right)^2}$$

- set $K = \alpha + \varepsilon$, for arbitrary small $\varepsilon > 0$: the assumption is satisfied if

$$K_P > 2\alpha + \frac{\alpha^2}{\varepsilon} \qquad \longrightarrow \qquad \text{for } \varepsilon \to 0 \quad \Rightarrow \quad K_P \to \infty$$

- in the presence of model uncertainties, the control law (with K_P large enough)

$$\tau = K_P(\hat{\theta}_d - \theta) - K_D\dot{\theta} + \hat{g}(q_d) \qquad \hat{\theta}_d := q_d + \hat{K}^{-1}\hat{g}(q_d)$$

provides asymptotic stability, for a different (still unique) equilibrium $(\bar{q}, \bar{\theta})$

- a version with on-line gravity compensation (De Luca, Siciliano, Zollo, 2005)

$$\tau = K_P(\theta_d - \theta) - K_D\dot{\theta} + g(\tilde{\theta})$$

where $\tilde{\theta} := \theta - K^{-1}g(q_d)$ is a 'biased' motor position measurement

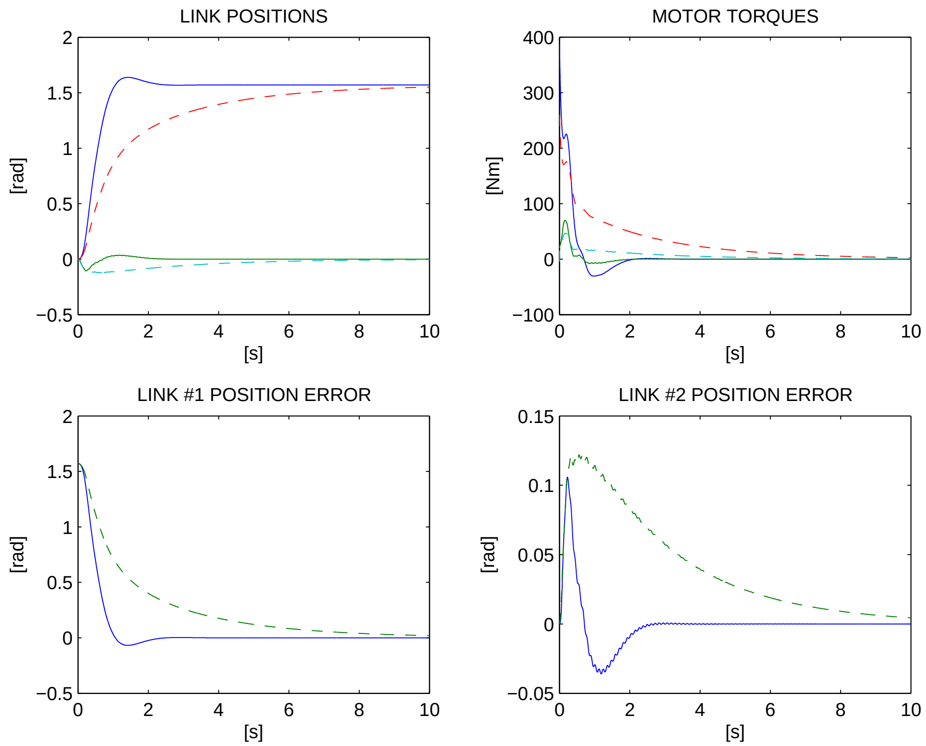
  - proof uses a modified Lyapunov candidate with

$$P' = \frac{1}{2}(q - \theta)^T K(q - \theta) + \frac{1}{2}(\theta_d - \theta)^T K_P(\theta_d - \theta) + U_g(q) - U_g(\tilde{\theta})$$

- however, the available proof does not relax the assumptions on a minimum K (structural) and on the need of an associated lower bound involving K_P (minimum positional control gain)

Comparative simulation results on a 2R robot

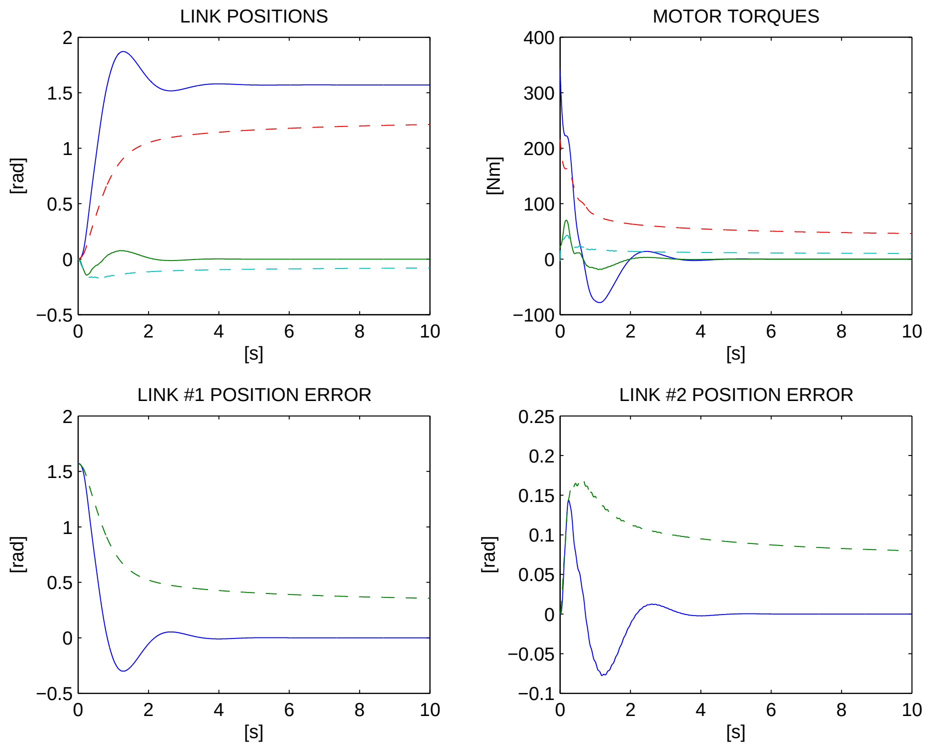
$$K_P = \text{diag}\{180, 180\} \quad K_D = \text{diag}\{80, 80\} \qquad (\alpha \simeq 133)$$



on-line (solid) vs. constant (dashed) gravity compensation

Comparative simulation results on a 2R robot

$K_P = \text{diag}\{150, 150\}$ $\quad K_D = \text{diag}\{50, 50\}$ (sufficiency is violated)



on-line (solid) vs. constant (dashed) gravity compensation

Further remarks on regulation control

- a stronger result is obtained using on-line quasi-static gravity cancellation (Kugi, Ott, Albu-Schäffer, Hirzinger, 2008)

$$\tau = K_P(\theta_d - \theta) - K_D\dot{\theta} + g(\bar{q})$$

where $\bar{q} = \bar{q}(\theta)$ is obtained by solving iteratively, at any given position θ

$$g(\bar{q}) + K(\bar{q} - \theta) = 0 \quad \Rightarrow \quad q^i = \theta - K^{-1} g(q^{i-1})$$

- the sequence $\{q^0 = \theta, q^1, q^2, \ldots\}$ converges (in few iterations) to $\bar{q}$ thanks to a contraction mapping result $\Rightarrow$ structural assumption $\min_{i=1,\ldots,N} K_i > \alpha$ is kept, while any $K_P > 0$ is sufficient
- an even stronger result can be obtained using a nonlinear PD law, including dynamic gravity cancellation on the link dynamics (De Luca, Flacco, 2011), based on the feedback equivalence principle $\Rightarrow$ any $K > 0$ and $K_P > 0$ will be sufficient

Trajectory tracking

- assuming that
  - $q_d(t) \in C^4$ (fourth derivative w.r.t. time exists)
  - full state is measurable

  we proceed by system inversion from the link position output q

- a nonlinear static state feedback will be obtained that decouples and exactly linearizes the closed-loop dynamics (set of in-out integrators) for any value K

- exponential stabilization of the tracking error is then performed on the linear side of the transformed problem

$$\Downarrow$$

a generalized computed torque law

- original result (Spong, 1987), revisited without the need of state-space concepts

Feedback linearization by system inversion

- differentiate the link equation of the dynamic model (independent of input τ)

$$M(q)\ddot{q} + n(q,\dot{q}) + K(q-\theta) = 0 \qquad n(q,\dot{q}) := C(q,\dot{q})\dot{q} + g(q)$$

to obtain (notation: $q^{[i]} = d^i q/dt^i$)

$$M(q)q^{[3]} + \dot{M}(q)\ddot{q} + \dot{n}(q,\dot{q}) + K(\dot{q}-\dot{\theta}) = 0$$

still independent from τ

- differentiating once more (fourth derivative of q appears)

$$M(q)q^{[4]} + 2\dot{M}(q)q^{[3]} + \ddot{M}(q)\ddot{q} + \ddot{n}(q,\dot{q}) + K(\ddot{q}-\ddot{\theta}) = 0$$

the input τ appears through $\ddot{\theta}$ and the motor equation

$$B\ddot{\theta} + K(\theta - q) = \tau$$

- substitution of $\ddot{\theta}$ gives

$$M(q)q^{[4]} + c(q, \dot{q}, \ddot{q}, q^{[3]}) + KB^{-1}K(\theta - q) = KB^{-1}\tau$$

with

$$c(q, \dot{q}, \ddot{q}, q^{[3]}) := 2\dot{M}(q)q^{[3]} + (\ddot{M}(q) + K)\ddot{q} + \ddot{n}(q, \dot{q})$$

- the control law

$$\tau = BK^{-1}\left(M(q)a + c(q, \dot{q}, \ddot{q}, q^{[3]})\right) + K(\theta - q)$$

leads to the closed-loop system

$$\boxed{q^{[4]} = a}$$

N separate input-output chains of four integrators (linearization and decoupling)

- $(q, \dot{q}, \ddot{q}, q^{[3]})$ is an alternative globally defined state representation
  - from the link equation

$$\ddot{q} = M^{-1}(q)\left(K(\theta - q) - n(q, \dot{q})\right)$$

    i.e., link acceleration is a function of $(q, \theta, \dot{q})$
  - from the first differentiation of the link equation

$$q^{[3]} = M^{-1}(q)\left(K(\dot{\theta} - \dot{q}) - \dot{M}(q)\ddot{q} - \dot{n}(q, \dot{q})\right)$$

    i.e., link jerk is a function of $(q, \theta, \dot{q}, \dot{\theta})$ (using the above expression for $\ddot{q}$)
  - the control term $c(q, \dot{q}, \ddot{q}, q^{[3]})$ can be expressed as a function of $(q, \theta, \dot{q}, \dot{\theta})$, with an efficient organization of computations
- the control law $\tau = \tau(q, \theta, \dot{q}, \dot{\theta}, a)$ can be implemented as a nonlinear static (instantaneous) feedback from the original state

Tracking error stabilization

- control design is completed on the linear side of the problem by choosing

$$a = q_d^{[4]} + K_3(q_d^{[3]} - q^{[3]}) + K_2(\ddot{q}_d - \ddot{q}) + K_1(\dot{q}_d - \dot{q}) + K_0(q_d - q)$$

with $\ddot{q}$ and $q^{[3]}$ expressed in terms of $(q, \theta, \dot{q}, \dot{\theta})$ and diagonal gain matrices $K_0, \ldots, K_3$ chosen such that

$$s^4 + K_{3i}s^3 + K_{2i}s^2 + K_{1i}s + K_{0i} \qquad i = 1, \ldots, N$$

are Hurwitz polynomials

- the tracking errors $e_i = q_{di} - q_i$ on each link coordinate satisfy

$$e_i^{[4]} + K_{3i}e_i^{[3]} + K_{2i}\ddot{e}_i + K_{1i}\dot{e}_i + K_{0i}e_i = 0$$

i.e., exponentially stable linear differential equations (with chosen eigenvalues)

Remarks on trajectory tracking control

- the shown feedback linearization result is the nonlinear/MIMO counterpart of the transfer function $\tau \longrightarrow q$ being without zeros (no zero dynamics)
- the same result can be rephrased as "q is a flat output for EJ robots"
- a nominal feedforward torque ($\equiv$ inverse dynamics!) can be computed off line

$$\tau_d = B\ddot{\theta}_d + K(\theta_d - q_d)$$

using the previous developments, where

$$K(\theta_d - q_d) = M(q_d)\ddot{q}_d + n(q_d, \dot{q}_d) \quad \ddot{\theta}_d = K^{-1}\left(M(q_d)q_d^{[4]} + c(q_d, \dot{q}_d, \ddot{q}_d, q_d^{[3]})\right)$$

- a simpler tracking controller (of local validity around the reference trajectory) is

$$\tau = \tau_d + K_P(\theta_d - \theta) + K_D(\dot{\theta}_d - \dot{\theta})$$

Two-time scale control design

- for high stiffness K the system is singularly perturbed $\Rightarrow$ may use a simpler composite control law, combining a slow and a fast component

$$\tau = \tau_s(q, \dot{q}, t) + \epsilon \tau_f(q, \dot{q}, z, \dot{z}, t)$$

where $\tau_s = \tau|_{\epsilon=0}$ depends only on the slow dynamics of link motion (time t in the arguments may come from the reference trajectory $q_d(t)$ to be tracked)

- the slow control τ_s is designed on the equivalent rigid dynamics (e.g., a feedback linearization/computed torque or a PD law) obtained by setting $\epsilon = 0$ in the singularly perturbed model, whereas the fast control τ_f is used for stabilization of fast dynamics due to elasticity around the manifold of equivalent rigid motion
- the control design is thus split in two parts that work at different time scales: we should avoid to mix back these through feedback (τ_f should not contain terms of order $1/\epsilon$ or higher)

- use the input $\tau = \tau_s + \epsilon\tau_f$ in the fast dynamics of the singularly perturbed model (see slide 20), set $\epsilon = 0$ (in the limit), and solve for z

$$z = \left(B^{-1} + M^{-1}(q)\right)\left(B^{-1}\tau_s + M^{-1}(q)\left(C(q,\dot{q})\dot{q} + g(q)\right)\right) \qquad (*)$$
$$= h(q, \dot{q}, \tau_s(q, \dot{q}, t))$$

a control-dependent manifold in the $4N$-dimensional state space of the robot

- replacing in the slow dynamics ($M(q)\ddot{q} + \ldots = z$) yields, after simplifications

$$(M(q)+B)\,\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau_s$$

slow reduced ($2N$-dim) system

which is the equivalent rigid robot dynamics (obtained for $K \to \infty$!)

- for tracking a reference trajectory $q_d(t) \in C^2$, choose, e.g., a slow control law based on feedback linearization

$$\tau_s = (M(q) + B)\left(\ddot{q}_d + K_D(\dot{q}_d - \dot{q}) + K_P(q_d - q)\right) + C(q,\dot{q})\dot{q} + g(q)$$
$$= \tau_s(q, \dot{q}, t)$$

- substitute the (partially designed) control law $\tau = \tau_s(q, \dot{q}, t) + \epsilon\tau_f$ in the fast dynamics of the singularly perturbed model

$$\epsilon^2 \ddot{z} = \hat{K}\left(B^{-1}\left(\tau_s(q, \dot{q}, t) + \epsilon\tau_f\right) - \left(B^{-1} + M^{-1}(q)\right) z \right.$$
$$\left. + M^{-1}(q)\left(C(q, \dot{q})\dot{q} + g(q)\right)\right)$$

- due to time scale separation, the slow variables in the fast dynamics are assumed to stay constant to their values at $t = \bar{t}$ ($q = q(\bar{t}) = \bar{q}$, $\dot{q} = \dot{q}(\bar{t}) = \bar{\dot{q}}$), so

$$\epsilon^2 \ddot{z} = \hat{K}\left(B^{-1}\epsilon\tau_f - \left(B^{-1} + M^{-1}(\bar{q})\right) z\right) + w(\bar{q}, \bar{\dot{q}}, \bar{t})$$

where

$$w(\bar{q}, \bar{\dot{q}}, \bar{t}) = \hat{K}\left(B^{-1}\tau_s(\bar{q}, \bar{\dot{q}}, \bar{t}) + M^{-1}(\bar{q})\left(C(\bar{q}, \bar{\dot{q}})\bar{\dot{q}} + g(\bar{q})\right)\right)$$

which in turn, when compared with the manifold defined by $(*)$, is rewritten as

$$w(\bar{q}, \bar{\dot{q}}, \bar{t}) = \hat{K}\left(B^{-1} + M^{-1}(\bar{q})\right)\bar{z}$$

$\Rightarrow$ $\bar{z}$ will be treated as a constant parameter in the fast dynamics

- defining the error on the fast variables as $\zeta = z - \bar{z}$, its dynamics is

$$\epsilon^2 \ddot{\zeta} \; (= \epsilon^2 \ddot{z}) \; = \; \widehat{K} \left(B^{-1} \epsilon \tau_f + \left(B^{-1} + M^{-1}(\bar{q}) \right) (\bar{z} - z) \right) \\ = \; \widehat{K} \left(B^{-1} \epsilon \tau_f - \left(B^{-1} + M^{-1}(\bar{q}) \right) \zeta \right)$$

- the fast control law should stabilize this linear error dynamics so that the fast variable z converges to its boundary layer $\bar{z}$
- with a diagonal $K_f > 0$ (but such that $\lambda_{max}(K_f) \ll \frac{1}{\epsilon}$), the choice

$$\tau_f = -K_f \dot{\zeta} = \tau_f(q, \dot{q}, z, \dot{z}, t)$$

leads to the exponentially stable error dynamics

$$\epsilon^2 \ddot{\zeta} + \left(\widehat{K} B^{-1} K_f \right) \epsilon \dot{\zeta} + \left(\widehat{K} \left(B^{-1} + M^{-1}(\bar{q}) \right) \right) \zeta = 0$$

(being all matrices positive definite)

- the final control law is $\tau = \tau_s(q, \dot{q}, t) - \epsilon K_f \dot{z}$

An extension – Invariant manifold control design

- in the previous analysis, the slow control component τ_s works correctly, i.e., it tracks the reference trajectory $q_d(t)$ on the equivalent rigid manifold, only for $\epsilon = 0$
- to improve the local behavior around an equivalent reduced ($2N$-dim) manifold in the $I\!R^{4N}$ state space, we add corrective terms

$$\tau_s = \tau_0 + \epsilon\tau_1 + \epsilon^2\tau_2 + \ldots$$

(in the previous control law, $\tau_0 = \tau_s$)

- proceed as before for the slow control design, but using a similar expansion of the resulting manifold (compare with $(*)$)

$$\begin{aligned} z &= h(q, \dot{q}, z, \dot{z}, \epsilon, t) \\ &= h_0(q, \dot{q}, z, \dot{z}, t) + \epsilon h_1(q, \dot{q}, z, \dot{z}, t) + \epsilon^2 h_2(q, \dot{q}, z, \dot{z}, t) + \ldots \end{aligned}$$

- in particular, by using up to the second-order correction term, it can be shown that the resulting manifold can be made invariant

  if the initial state starts on the (integral) manifold, the controlled trajectories will remain on it —unless disturbances occur

- this result is similar to the one obtained by feedback linearization, but restricted to the integral manifold
- the fast control law is then needed only for recovering from initial state mismatch and/or disturbances
- see (Spong, Khorasani, Kokotovic, 1987)

Robots with mixed rigid/elastic joints

- consider an N-dof robot having R rigid joints, characterized by $q_r \in I\!R^R$, and $N-R$ elastic joints, characterized by link variables $q_e \in I\!R^{N-R}$ and motor variables $\theta_e \in I\!R^{N-R} \Rightarrow$ the state dimension is $2R + 4(N-R) = 4N - 2R$
- under assumption A4), the dynamic model can be rewritten in a partitioned way (possibly, after reordering of joint variables) as

$$\begin{pmatrix} M_{rr}(q) & M_{re}(q) \\ M_{re}^T(q) & M_{ee}(q) \end{pmatrix} \begin{pmatrix} \ddot{q}_r \\ \ddot{q}_e \end{pmatrix} + \begin{pmatrix} n_r(q,\dot{q}) \\ n_e(q,\dot{q}) \end{pmatrix} + \begin{pmatrix} 0 \\ K_e(q_e - \theta_e) \end{pmatrix} = \begin{pmatrix} \tau_r \\ 0 \end{pmatrix}$$

$$B_e\ddot{\theta}_e + K_e(\theta_e - q_e) = \tau_e$$

where $q=(q_r^T\ q_e^T)^T \in I\!R^N$, the $2N \times 2N$ inertia matrix $M(q)$ and its diagonal blocks $M_{rr}(q)$ and $M_{ee}(q)$ are invertible for all q, the $2N$-vector $n(q,\dot{q})=(n_r^T(q,\dot{q})\ n_e^T(q,\dot{q}))^T$ collects all centrifugal/Coriolis and gravity terms, K_e is the diagonal $(N-R)\times(N-R)$ stiffness matrix of the elastic joints, and $\tau=(\tau_r^T\ \tau_e^T)^T \in I\!R^N$ are the input torques

- for the link accelerations (i.e., applying the system inversion algorithm to the output $y = q$, after two time derivatives)

$$\begin{pmatrix} \ddot{q}_r \\ \ddot{q}_e \end{pmatrix} = \begin{pmatrix} \left(M_{rr} - M_{re}M_{ee}^{-1}M_{re}^T\right)^{-1} & 0 \\ \left(M_{ee} - M_{re}^T M_{rr}^{-1} M_{re}\right)^{-1} M_{re}^T M_{rr}^{-1} & 0 \end{pmatrix} \begin{pmatrix} \tau_r \\ 0 \end{pmatrix}$$

$$+ \begin{pmatrix} \gamma_r(q, \dot{q}, \theta_e) \\ \gamma_e(q, \dot{q}, \theta_e) \end{pmatrix} = \mathcal{A}(q)\tau + \gamma(q, \dot{q}, \theta_e)$$

- it is easy to see that $\mathcal{A}(q)$ is the decoupling matrix of the system (i.e., all its rows should be non-zero) as soon as $M_{re} \not\equiv 0$
- if this is the case, $\mathcal{A}(q)$ is always singular (due to the last columns of zeros) $\Rightarrow$ the necessary (and sufficient) condition for input-output decoupling by static state feedback is violated
- thus, if $M_{re} \not\equiv 0$, the more general class of dynamic state feedback may be needed for obtaining exact linearization of the closed-loop system

- consider then the case $M_{re} \equiv 0$; moreover, assume that $n_e = n_e(q, \dot{q}_e)$ (i.e., is independent from $\dot{q}_r$)
- this happens if and only if $M_{rr} = M_{rr}(q_r)$, $M_{ee} = M_{ee}(q_e)$ (using the Christoffel symbols for the derivation of the Coriolis/centrifugal terms from the robot inertia matrix)
- the latter implies also $n_r = n_r(q, \dot{q}_r) \Rightarrow$ a complete inertial separation between the dynamics of the rigidly driven and of the elastically driven links follows

$$\Rightarrow \quad \begin{cases} M_{rr}(q_r)\ddot{q}_r + n_r(q, \dot{q}_r) = \tau_r \\ M_{ee}(q_e)\ddot{q}_e + n_e(q, \dot{q}_e) + K_e(q_e - \theta_e) = 0 \\ B_e\ddot{\theta}_e + K(\theta_e - q_e) = \tau_e \end{cases}$$

Theorem 1 (De Luca, 1998) Robots having mixed rigid/elastic joints can be input-output decoupled (with $y = q$) and exactly linearized by static state feedback if and only if there is complete inertial separation in the structure, i.e.

1. $M_{re} \equiv 0$
2. $M_{rr} = M_{rr}(q_r)$, $M_{ee} = M_{ee}(q_e)$

The resulting closed-loop linear system is in the form $\ddot{q}_r = a_r$, $q_e^{[4]} = a_e$

Under the hypotheses of the Theorem, the feedback linearization control law is

$$\tau_r = M_{rr}(q_r)a_r + n_r(q, \dot{q}_r) \qquad \tau_e = BK^{-1}\left(M_{ee}(q_e)a_e + c_e(q, \dot{q}, \ddot{q}_e, q_e^{[3]})\right)$$

where $c_e(\cdot) := 2\dot{M}_{ee}(q_e)q_e^{[3]} + (\ddot{M}_{ee}(q_e) + K_e)\ddot{q}_e + \ddot{n}_e(q, \dot{q}_e)$ and

$$\begin{aligned} \ddot{q}_e &= M_{ee}^{-1}(q_e)\left(K_e(\theta_e - q_e) - n_e(q, \dot{q}_e)\right) \\ q_e^{[3]} &= M_{ee}^{-1}(q_e)\left(K_e(\dot{\theta}_e - \dot{q}_e) - \dot{M}_{ee}(q_e)\ddot{q}_e - \dot{n}_e(q, \dot{q}_e)\right) \end{aligned}$$

Theorem 2 (De Luca, 1998) When Theorem 1 cannot be applied, robots having mixed rigid/elastic joints can always be input-output decoupled (with $y = q$) and exactly linearized by a dynamic state feedback law of order $m = 2R$. The resulting closed-loop linear system is in the form $q_r^{[4]} = a_r$, $q_e^{[4]} = a_e$

The following linear dynamic compensator, with state $\xi = (\theta_r^T\ \dot{\theta}_r^T)^T \in I\!R^{2R}$

$$B_r\ddot{\theta}_r + K_r(\theta_r - q_r) = \tau_{re} \qquad \tau_r = K_r(\theta_r - q_r)$$

having arbitrary (diagonal) $B_r > 0$, $K_r > 0$ and new input $\tau_{re} \in I\!R^R$, extends the original mixed rigid/elastic dynamics to the same structure with all elastic joints

$$\begin{pmatrix} M_{rr}(q) & M_{re}(q) \\ M_{re}^T(q) & M_{ee}(q) \end{pmatrix} \begin{pmatrix} \ddot{q}_r \\ \ddot{q}_e \end{pmatrix} + \begin{pmatrix} n_r(q,\dot{q}) \\ n_e(q,\dot{q}) \end{pmatrix} + \begin{pmatrix} K_r(q_r-\theta_r) \\ K_e(q_e-\theta_e) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} B_r & 0 \\ 0 & B_e \end{pmatrix} \begin{pmatrix} \ddot{\theta}_r \\ \ddot{\theta}_e \end{pmatrix} + \begin{pmatrix} K_r(\theta_r-q_r) \\ K_e(\theta_e-q_e) \end{pmatrix} = \begin{pmatrix} \tau_{re} \\ \tau_e \end{pmatrix}$$

$\Rightarrow$ feedback linearizable by a static feedback from the extended state ...

A more complete dynamic model of EJ robots

- what happens if we remove the simplifying assumption A3)?
- for a planar 2R EJ robot, the complete angular kinetic energy of the motors is

$$T_{m1} = \frac{1}{2} J_{m1} r_1^2 \dot{\theta}_1^2 \qquad T_{m2} = \frac{1}{2} J_{m2} (\dot{q}_1 + r_2 \dot{\theta}_2)^2$$

with no changes at base motor and new terms for elbow motor; as a result

$$\begin{aligned} T_m &= \tfrac{1}{2} \begin{pmatrix} \dot{q}^T & \dot{\theta}^T \end{pmatrix} \begin{pmatrix} J_{m2} & 0 & 0 & J_{m2} r_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & J_{m1} r_1^2 & 0 \\ J_{m2} r_2 & 0 & 0 & J_{m2} r_2^2 \end{pmatrix} \begin{pmatrix} \dot{q} \\ \dot{\theta} \end{pmatrix} \\ &= \tfrac{1}{2} \begin{pmatrix} \dot{q}^T & \dot{\theta}^T \end{pmatrix} \begin{pmatrix} \begin{matrix} J_{m2} & 0 \\ 0 & 0 \end{matrix} & S \\ S^T & B \end{pmatrix} \begin{pmatrix} \dot{q} \\ \dot{\theta} \end{pmatrix} \end{aligned}$$

the blue terms contribute to $M(q)$ (the diagonal 0 should not worry here!)

- for NR planar EJ robots, we obtain

$$\begin{aligned} M(q)\ddot{q} + S\ddot{\theta} + C(q,\dot{q})\dot{q} + g(q) + K(q-\theta) &= 0 \\ S^T\ddot{q} + B\ddot{\theta} + K(\theta - q) &= \tau \end{aligned}$$

link equation

motor equation

with the strictly upper triangular matrix S representing inertial couplings between motor and link accelerations

- in general, a non-constant matrix S may arise, see (De Luca, Tomei, 1996)

$$S(q) = \begin{bmatrix} 0 & S_{12}(q_1) & S_{13}(q_1,q_2) & \cdots & S_{1N}(q_1,\ldots,q_{N-1}) \\ 0 & 0 & S_{23}(q_2) & \cdots & S_{2N}(q_2,\ldots,q_{N-1}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & S_{N-1,N}(q_{N-1}) \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

with new associated centrifugal/Coriolis terms in both link and motor equations

Control-oriented remarks

- specific kinematic structures with elastic joints (single link, polar 2R, PRP, ...) have $S \equiv 0$, so that the reduced model is also complete and feedback linearizable
- for the inverse dynamics solution, see (De Luca, Book, 2008)
- as soon as $S \neq 0$, exact linearization and input-output decoupling both fail if we rely on the use of a nonlinear but static state feedback structure
- in order to mimic a generalized computed torque approach (linearization and decoupling for tracking tasks), we need a dynamic state feedback controller

$$\begin{aligned} \tau &= \alpha(x,\xi) + \beta(x,\xi)v \\ \dot{\xi} &= \gamma(x,\xi) + \delta(x,\xi)v \end{aligned}$$

with robot state $x = (q, \theta, \dot{q}, \dot{\theta}) \in I\!R^{4N}$, dynamic compensator state $\xi \in I\!R^m$ (yet to be defined), and new control input $v \in I\!R^N$

A control extension — Dynamic feedback linearization of EJ robots

- a three-step design that achieves full linearization and input-output decoupling, incrementally building the compensator dynamics through the solution of two intermediate subproblems $\Rightarrow$ DFL algorithm in (De Luca, Lucibello, 1998)
- presented for constant $S \neq 0$, works also for $S(q)$
- transformation of the dynamic equations in state-space format is not needed
- collapses in the usual linearizing control by static state feedback when $S = 0$
- can be applied also to the complete model of robots with joints of mixed type —some rigid, other elastic, see (De Luca, Farina, 2004)

Step 1: I-O decoupling w.r.t. θ

- apply the *static* control law

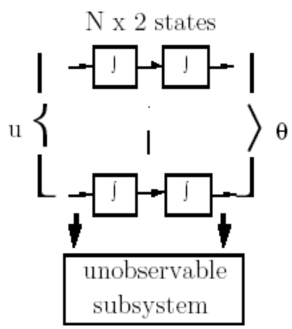
$$\tau = Bu + S^T \ddot{q} + K(\theta - q)$$

or, eliminating link acceleration $\ddot{q}$ (and dropping dependencies)

$$\tau = \left(J - S^T M^{-1} S\right) u - S^T M^{-1} \left(C\dot{q} + g + K(q - \theta)\right) + K(\theta - q)$$

- the resulting system is

$$\begin{aligned} \ddot{\theta} &= u \\ M(q)\ddot{q} &= \ldots \text{ (}2N \text{ dynamics unobservable from } \theta\text{)} \end{aligned}$$



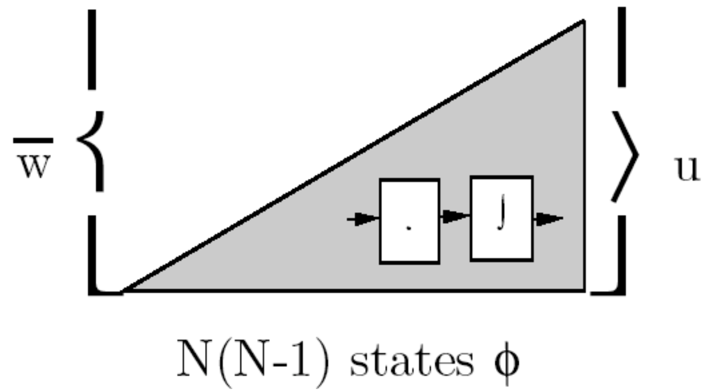
Step 2: I-O decoupling w.r.t. f

- define as output the generalized force

$$f = M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) + Kq$$

$\Rightarrow$ the link equation, after Step 1, is rewritten as $f(q,\dot{q},\ddot{q}) + Su - K\theta = 0$

- *dynamic extension:* add $2(i-1)$ integrators on input u_i $(i = 1, \ldots, N)$ so as to avoid successive input differentiation

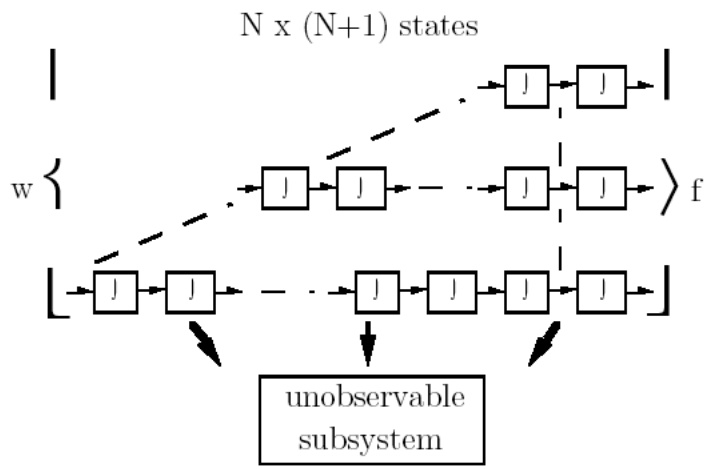


- differentiate $2i$ times the component f_i $(i = 1, \ldots, N)$ and apply a linear *static* control law (depending on K and S)

$$\overline{w} = F_w \phi + G_w w$$

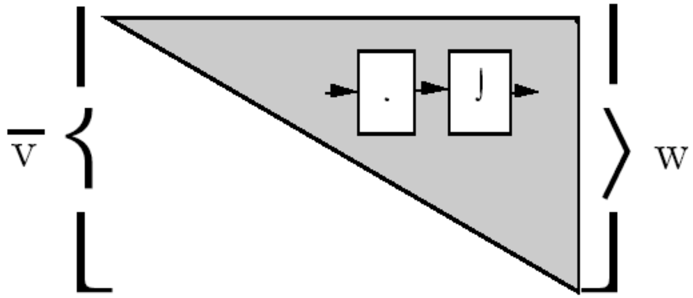
so that the resulting input-output system is

$$\frac{d^{2i} f_i}{dt^{2i}} = w_i \qquad i = 1, \ldots, N$$



Step 3: I-O decoupling w.r.t. q

- *dynamic balancing:* add $2(N-i)$ integrators on input w_i ($i = 1, \ldots, N$) so as to avoid successive input differentiation



N(N-1) states ψ

- differentiate $2(N-i)$ times the ith input-output equation ($i = 1, \ldots, N$) after Step 2, thus obtaining

$$\frac{d^{2N} f}{dt^{2N}} = \frac{d^{2N}}{dt^{2N}}(M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) + Kq) = \bar{v}$$

- apply the nonlinear *static* control law (globally defined)

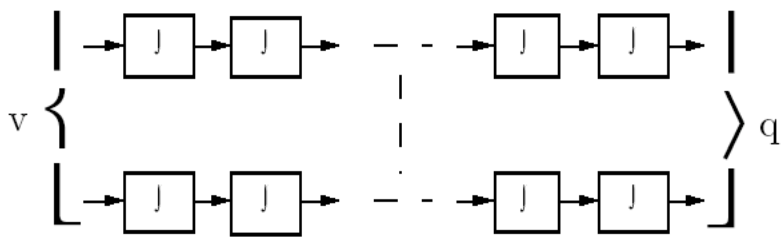
$$\overline{v} = M(q)v + \tilde{n}(q, \dot{q}, \ldots, q^{[2N+1]}) + g^{[2N]}(q) + Kq^{[2N]}$$

where

$$\tilde{n} = \sum_{k=1}^{2N} \binom{2N}{k} M^{[k]}(q) q^{[2(N+1)-k]} + \sum_{k=0}^{2N} \binom{2N}{k} C^{[k]}(q, \dot{q}) q^{[2N+1-k]}$$

- the final resulting system is fully linearized and decoupled

$$q^{[2(N+1)]} = v$$



N x 2(N+1) states

Comments on the DFL algorithm

- using recursion, the output q and all its derivatives *(linearizing coordinates)* can be expressed in terms of the robot + compensator states

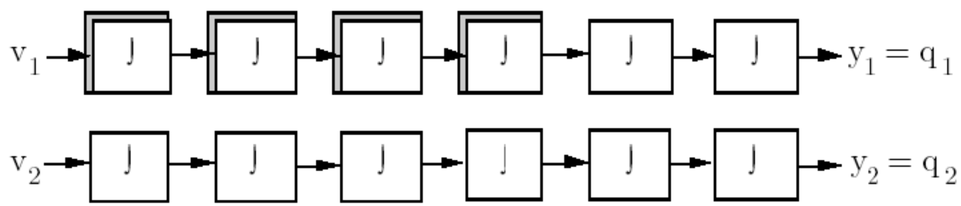
$$[\, q \quad \dot{q} \quad \ddot{q} \quad q^{[3]} \quad \dots \quad q^{[2N+1]} \,] = T(q, \theta, \dot{q}, \dot{\theta}, \phi, \psi)$$

- the overall nonlinear *dynamic feedback* for the original torque

$$\tau = \tau(q, \theta, \dot{q}, \dot{\theta}, \xi, v) \qquad \dot{\xi} = \begin{bmatrix} \dot{\phi} \\ \dot{\psi} \end{bmatrix} = \dots$$

is of dimension $m = 2N(N-1)$

- for a planar 2R EJ robot, $m = 4$ and final system with 2 chains of 6 integrators



- when some of the elements in the upper triangular part of S are zero (depending on the arm kinematics), then the needed dynamic controller has a dimension m that is lower than $2N(N-1)$ $\Rightarrow$ the dynamic extensions at steps 2 and 3 are required only for some joints
- for trajectory tracking purposes, given a (sufficiently smooth) $q_d(t) \in C^{2(N+1)}$, the tracking error $e_i = q_{di} - q_i$ on each channel is exponentially stabilized by

$$v_i = q_{di}^{[2(n+1)]} + \sum_{j=0}^{2N+1} K_{ji} \left(q_{di}^{[j]} - q_i^{[j]} \right) \qquad i = 1, \ldots, N$$

where $K_{0i}, \ldots, K_{2N+1,i}$ are the coefficients of a desired Hurwitz polynomial

Final remarks

- for the complete dynamic model of EJ robots, all proposed control laws for regulation tasks are still valid
  - under the same conditions, using the same Lyapunov candidates in the proof, with a more complex final LaSalle analysis
- addition of viscous friction terms on the lhs of the link ($D_q\dot{q}$) and the motor ($D_\theta\dot{\theta}$) equations, with diagonal $D_q, D_\theta > 0$, is trivially handled both in regulation and trajectory tracking controllers
- inclusion of spring damping ($+D(\dot{q} - \dot{\theta})$ on the lhs of the link equation, its opposite in the motor equation) $\Rightarrow$ visco-elastic joints
  - essentially, no changes for regulation controllers
  - static feedback linearization for tracking tracking tasks becomes ill-conditioned for $D \to 0$, while resorting to a dynamic feedback linearization control will guarantee regularity (De Luca, Farina, Lucibello, 2005)

Research issues

- kinetostatic calibration of EJ robots using only motor measurements
- unified dynamic identification of model parameters (including K and B)
- robust control for trajectory tracking in the presence of uncertainties
- adaptive control: available (but quite complex) only for the reduced model with $S = 0$ (Lozano, Brogliato, 1992)
- Cartesian impedance control with proved stability, see (Zollo, Siciliano, De Luca, Guglielmelli, Dario, 2005)
- fitting the results into parallel/redundant actuated devices with joint elasticity
- consideration of nonlinear transmission flexibility, with passively varying stiffness or independently actuated
- ...

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