Robotics 1

Midterm Test — November 22, 2024

Exercise 1

The end-effector of a robot manipulator has an initial orientation specified by the ZXY Euler angles $(\alpha, \beta, \gamma) = (\pi/2, \pi/4, -\pi/4)$ [rad] and should reach a final orientation specified by an axis-angle pair $(\boldsymbol{r}, \theta)$, with $\boldsymbol{r} = (0, -\sqrt{2}/2, \sqrt{2}/2)$ and $\theta = \pi/6$ rad. What is the required rotation matrix $\boldsymbol{R}_{if}$ between these two orientations? Represent $\boldsymbol{R}_{if}$ by the RPY-type angles (ϕ, χ, ψ) around the fixed-axes sequence YXY.

Exercise 2

A cylinder of height h and radius r lies on the plane (x_w, y_w) in the initial pose shown in Fig. 1, with a frame $RF_c = (x_c, y_c, z_c)$ attached to the geometric center of its body. The cylinder rolls without slipping by a ground distance $d > 0$ in the y_w-direction, and rotates then by an angle ϑ around the original z_w-axis. Finally, a rotation φ is performed around the current direction of the z_c-axis. Determine the expression of the elements of the homogeneous transformation matrix ${}^w\boldsymbol{T}_c(h, r, d, \vartheta, \varphi)$ that characterizes the final pose of the cylinder. Evaluate then ${}^w\boldsymbol{T}_c$ for $h = 0.5$, $r = 0.1$, $d = 1.5$ [m] and $\vartheta = \pi/3$, $\varphi = -\pi/2$ [rad]. *Hint: Check your intermediate results with simpler data.*

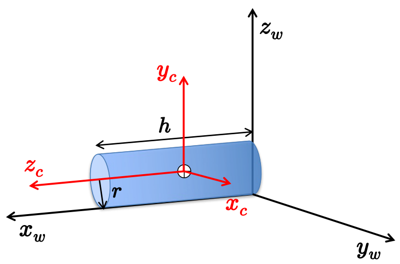

Figure 1: The initial set-up of a cylinder in the world frame.

Exercise 3

Consider the PPR planar robot with a 2-jaw gripper in Fig. 2, shown together with the world frame RF_w.

- Assign the link frames and fill in the associated table of parameters according to the Denavit–Hartenberg (DH) convention (use the extra sheet). The origin of the last DH frame should be placed at the gripper's center (point P). Choose the frames so that there is **no** axis pointing inside the sheet.
- Determine the homogeneous transformation matrices ${}^w\boldsymbol{T}_0$ and ${}^3\boldsymbol{T}_e$, respectively between the world frame RF_w and the zero-th DH frame RF_0 and between the last DH frame RF_3 and the end-effector frame RF_e placed at the gripper, with the usual convention ($\boldsymbol{z}_e$ in the approach direction and $\boldsymbol{y}_e$ in the open/close slide direction of the jaws).
- Provide the direct kinematics for the end-effector position ${}^w\boldsymbol{p}_e \in \mathbb{R}^3$.
- When the two prismatic joints are limited as $q_i \in [q_{i,m}, q_{i,M}]$, under the assumption that $q_{i,M} - q_{i,m} > 2L$, for $i = 1, 2$, and the revolute joint is in the range $q_3 \in [-3\pi/4, 0]$, sketch the primary workspace of this robot and locate the relevant points on its boundary.

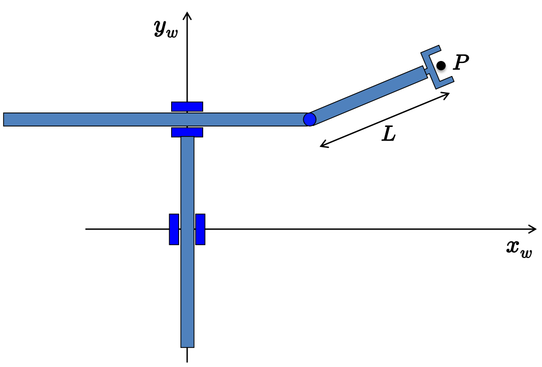

Figure 2: A PPR planar robot with last link of length L.

Exercise 4

With reference to the scheme in Fig. 3, assume that the three toothed gears of the transmission have radius, respectively, $r_m = 0.5$, $r_e = 40$, and $r_l = 10$ [cm]. The motor inertia is $J_m = 7.1 \cdot 10^{-4}$ kgm^2, while the inertia of the link around its rotation axis is denoted by J_l. An incremental encoder is mounted on the axis of the middle gear. Gravity is absent and inertia and friction of the gears are negligible.

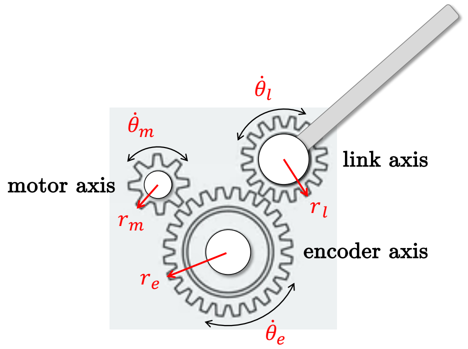

Figure 3: Transmission gears from motor to link, using an incremental encoder.

- What is the value of the link inertia J_l that optimizes torque transmission?
- With this J_l, what is the acceleration $\ddot{\theta}_l$ when the motor delivers on its axis a torque $\tau_m = 10$ [Nm]?
- For a link resolution of 0.01°, how many pulses per turn (with quadrature) should the encoder have?
- With this resolution, what is the average speed $\dot{\theta}_m$ when the encoder increments 100 pulses per second?

Exercise 5

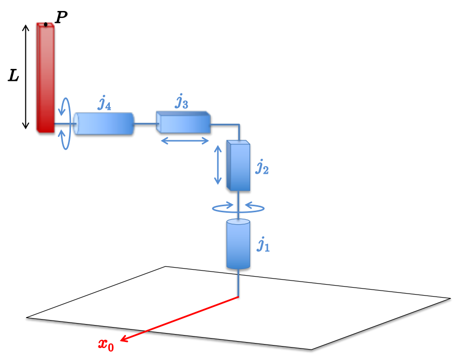

Figure 4: An RPPR spatial robot.

i	α_i	a_i	d_i	θ_i
1	0	0	0	q_1
2	$\pi/2$	0	q_2	0
3	0	0	q_3	0
4	0	L	0	q_4

Table 1: D-H parameters of the RPPR robot.

The RPPR spatial robot shown in Fig. 4 has the DH parameters given in Tab. 1.

- Draw the corresponding DH frames (use the extra sheet) and give the values, or at least the signs, of the components of $\boldsymbol{q}$ in the shown configuration.
- Consider the task vector

$$\boldsymbol{r} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ \alpha \end{pmatrix} = \begin{pmatrix} \sin q_1 q_3 + L \cos q_1 \cos q_4 \\ -\cos q_1 q_3 + L \sin q_1 \cos q_4 \\ q_2 + L \sin q_4 \\ q_4 \end{pmatrix}. \tag{1}$$

  Solve the inverse kinematics problem in closed form for a given $\boldsymbol{r}_d \in \mathbb{R}^4$, determining also the possible singular situations. With $L = 1.5$ m, provide the numerical solutions for these data: $\boldsymbol{r}_{d1} = (2, 2, 4, -\pi/4)$, $\boldsymbol{r}_{d2} = (0, 0, 3, \pi/2)$, $\boldsymbol{r}_{d3} = (1, 1, 2, 0)$, and $\boldsymbol{r}_{d4} = (0, 1.5, 4, 0)$ [m,m,m,rad].

[180 minutes, open books]

Solution

November 22, 2024

Exercise 1

The initial orientation is specified by a ZXY Euler sequence (α, β, γ), which is associated to the rotation matrix

$$\boldsymbol{R}_{in}(\alpha,\beta,\gamma) \;=\; \boldsymbol{R}_{\boldsymbol{z}}(\alpha)\boldsymbol{R}_{\boldsymbol{x}}(\beta)\boldsymbol{R}_{\boldsymbol{y}}(\gamma) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\sin\alpha\cos\beta & \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma \\ \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\beta & \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma \\ -\cos\beta\sin\gamma & \sin\beta & \cos\beta\cos\gamma \end{pmatrix}.$$

When evaluated with the data, we obtain

$$\boldsymbol{R}_i = \boldsymbol{R}_{in}(\pi/2, \pi/4, -\pi/4) = \begin{pmatrix} 0.5000 & -0.7071 & 0.5000 \\ 0.7071 & 0 & -0.7071 \\ 0.5000 & 0.7071 & 0.5000 \end{pmatrix}.$$

On the other hand, the final orientation is given by the axis-angle method, with unit vector $\boldsymbol{r}$ and angle θ

$$\boldsymbol{R}_{fin}(\boldsymbol{r},\theta) = \boldsymbol{r}\boldsymbol{r}^T + \left(\boldsymbol{I} - \boldsymbol{r}\boldsymbol{r}^T\right)\cos\theta + \boldsymbol{S}(\boldsymbol{r})\sin\theta.$$

When evaluated with the data, we obtain

$$\boldsymbol{R}_f = \boldsymbol{R}_{fin}\left((0, -\sqrt{2}/2, \sqrt{2}/2), \pi/6\right) = \begin{pmatrix} 0.8660 & -0.3536 & -0.3536 \\ 0.3536 & 0.9330 & -0.0670 \\ 0.3536 & -0.0670 & 0.9330 \end{pmatrix}.$$

Therefore, the relative rotation to be realized is

$$\boldsymbol{R}_{if} = \boldsymbol{R}_i^T\boldsymbol{R}_f = \begin{pmatrix} 0.8598 & 0.4495 & 0.2424 \\ -0.3624 & 0.2026 & 0.9097 \\ 0.3598 & -0.8700 & 0.3371 \end{pmatrix}.$$

The RPY-type YXY sequence (ϕ, χ, ψ) is associated to the rotation matrix

$$\begin{aligned} \boldsymbol{R}_{if}(\phi,\chi,\psi) \;&=\; \boldsymbol{R}_{\boldsymbol{y}}(\psi)\boldsymbol{R}_{\boldsymbol{x}}(\chi)\boldsymbol{R}_{\boldsymbol{y}}(\phi) \\ &= \begin{pmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\chi & -\sin\chi \\ 0 & \sin\chi & \cos\chi \end{pmatrix} \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix} \\ &= \begin{pmatrix} \cos\phi\cos\psi - \sin\phi\cos\chi\sin\psi & \sin\chi\sin\psi & \sin\phi\cos\psi + \cos\phi\cos\chi\sin\psi \\ \sin\phi\sin\chi & \cos\chi & -\cos\phi\sin\chi \\ -\cos\phi\sin\psi - \sin\phi\cos\chi\cos\psi & \sin\chi\cos\psi & \cos\phi\cos\chi\cos\psi - \sin\phi\sin\psi \end{pmatrix}. \end{aligned}$$

The solution of the inverse problem for this sequence of angles

$$\boldsymbol{R}_{if}(\phi,\chi,\psi) = \boldsymbol{R}_{if}$$

is obtained from the expressions in the second row and second column of the above matrix. Denoting the numerical elements of matrix $\boldsymbol{R}_{if}$ by R_{ij}, under the regularity assumption

$$\sigma = \sqrt{R_{21}^2 + R_{23}^2} = |\sin\chi| \neq 0,$$

one obtains

$$\chi^{+,-} = \text{ATAN2}\left\{\pm\sqrt{R_{21}^2 + R_{23}^2}, R_{22}\right\},$$

and for each sign in this expression, the two pairs

$$\phi^+ = \text{ATAN2}\{R_{21}, -R_{23}\} \qquad \psi^+ = \text{ATAN2}\{R_{12}, R_{32}\}$$

and

$$\phi^- = \text{ATAN2}\{-R_{21}, R_{23}\} \qquad \psi^- = \text{ATAN2}\{-R_{12}, -R_{32}\}.$$

When substituting the numerical values, we find $\sigma = 0.9793$ and thus the two regular solutions

$$\begin{pmatrix} \phi^+ \\ \chi^+ \\ \psi^+ \end{pmatrix} = \begin{pmatrix} -2.7625 \\ 1.3668 \\ 2.6647 \end{pmatrix} \qquad \begin{pmatrix} \phi^- \\ \chi^- \\ \psi^- \end{pmatrix} = \begin{pmatrix} 0.3791 \\ -1.3668 \\ -0.4769 \end{pmatrix} \quad \text{[rad]}.$$

Finally, a convenient check of the correctness of the obtained results is to verify that

$$\boldsymbol{R}_i \boldsymbol{R}_{if}(\phi^+, \chi^+, \psi^+) \boldsymbol{R}_f^T = \boldsymbol{I},$$

and the same for the second solution (with the $-$ superscripts).

Exercise 2

The initial pose of the cylinder with respect to the world frame is

$${}^w\boldsymbol{T}_c^{in} = \begin{pmatrix} 0 & 0 & 1 & h/2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & r \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

When rolling the cylinder, a displacement $d > 0$ in the current x_c-direction corresponds to a clockwise rotation α around $\boldsymbol{z}_c$. Setting $\alpha = -d/r < 0$, the three elementary motions are described respectively by

$$\boldsymbol{T}_d = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & d \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \boldsymbol{T}_\vartheta = \begin{pmatrix} \cos\vartheta & -\sin\vartheta & 0 & 0 \\ \sin\vartheta & \cos\vartheta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \boldsymbol{T}_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which are combined as follows. The first rolling motion is defined with respect to the *current* pose ${}^w\boldsymbol{T}_c$ (namely, the initial one); thus

$$\boldsymbol{T}_1 = {}^w\boldsymbol{T}_c^{in}\,\boldsymbol{T}_d = \begin{pmatrix} 0 & 0 & 1 & h/2 \\ \cos\alpha & -\sin\alpha & 0 & d \\ \sin\alpha & \cos\alpha & 0 & r \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & h/2 \\ \cos(d/r) & \sin(d/r) & 0 & d \\ -\sin(d/r) & \cos(d/r) & 0 & r \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and one can see that the displacement $d > 0$ occurs in fact along the y_w-direction of the world frame. Moreover, if the distance travelled was $d = 2\pi r$ ($\alpha = -2\pi$, one full rotation), the orientation of the frame RF_c at the end would be again ${}^w\boldsymbol{R}_c$, as in the initial configuration, whereas for $d = \pi r/2$ ($\alpha = -\pi/2$, one fourth of a clockwise rotation), the axis x_c would be aligned with $-z_w$. The second rotation by ϑ occurs around the *fixed* axis z_w; thus, the order in the matrix product is

$$\boldsymbol{T}_2 = \boldsymbol{T}_\vartheta\,\boldsymbol{T}_1 = \boldsymbol{T}_\vartheta\,{}^w\boldsymbol{T}_c^{in}\,\boldsymbol{T}_d = \begin{pmatrix} -\sin\vartheta\cos\alpha & \sin\vartheta\sin\alpha & \cos\vartheta & (h/2)\cos\vartheta - d\sin\vartheta \\ \cos\vartheta\cos\alpha & -\cos\vartheta\sin\alpha & \sin\vartheta & d\cos\vartheta + (h/2)\sin\vartheta \\ \sin\alpha & \cos\alpha & 0 & r \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

One can easily check that a rotation by $\vartheta = \pi/2$ (counterclockwise around z_w) brings the position of the origin O_c to ${}^w\boldsymbol{p}_c = (-d, h/2, r)$, as expected. Finally, the third rotation by φ is defined again with respect to the *current* orientation of axis z_c; thus, we obtain the general symbolic expression

$$\boldsymbol{T}_3 = \boldsymbol{T}_2\,\boldsymbol{T}_\varphi = \boldsymbol{T}_\vartheta\,{}^w\boldsymbol{T}_c^{in}\,\boldsymbol{T}_d\,\boldsymbol{T}_\varphi$$

$$= \begin{pmatrix} -\cos\alpha\sin\vartheta\cos\varphi + \sin\alpha\sin\vartheta\sin\varphi & \cos\alpha\sin\vartheta\sin\varphi + \sin\alpha\sin\vartheta\cos\varphi & \cos\vartheta & (h/2)\cos\vartheta - d\sin\vartheta \\ \cos\alpha\cos\vartheta\cos\varphi - \sin\alpha\cos\vartheta\sin\varphi & -\sin\alpha\cos\vartheta\cos\varphi - \cos\alpha\cos\vartheta\sin\varphi & \sin\vartheta & (h/2)\sin\vartheta + d\cos\vartheta \\ \cos\alpha\sin\varphi + \sin\alpha\cos\varphi & \cos\alpha\cos\varphi - \sin\alpha\sin\varphi & 0 & r \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Substituting the given data, the final pose of the cylinder is

$${}^w\boldsymbol{T}_c = \boldsymbol{T}_{\vartheta=\pi/3}\;{}^w\boldsymbol{T}_c^{in}(h=0.5, r=0.1)\;\boldsymbol{T}_{d=1.5}\;\boldsymbol{T}_{\varphi=-\pi/2} = \begin{pmatrix} 0.5632 & 0.6579 & 0.5000 & -1.1740 \\ -0.3251 & -0.3798 & 0.8660 & 0.9665 \\ 0.7597 & -0.6503 & 0 & 0.1000 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Exercise 3

The assignment of DH frames for the PPR planar robot is shown in Fig. 5, with Tab. 2 containing the corresponding parameters. By following the given specifications, this assignment is unique up to the choice of the direction of the axis x_3 (which may instead point toward the axis of joint 3). Note that the third link points along y_w when $q_3 = 0$.

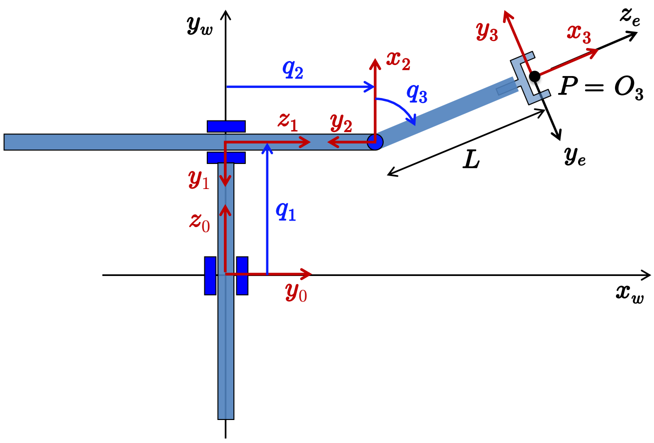

Figure 5: Assignment of DH frames for the PPR robot of Fig. 2.

i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	0	q_1	0
2	$-\pi/2$	0	q_2	$-\pi/2$
3	0	L	0	q_3

Table 2: DH parameters for the frame assignment in Fig. 5.

The two constant homogenous transformation matrices are

$$
{}^w\boldsymbol{T}_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad {}^3\boldsymbol{T}_e = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

Being $P = O_3 = O_e$, the direct kinematics for the end-effector position is extracted from

$$
{}^w\boldsymbol{p}_{e,hom} = {}^w\boldsymbol{T}_0 \, {}^0\boldsymbol{A}_1(q_1) \, {}^1\boldsymbol{A}_2(q_2) \, {}^2\boldsymbol{A}_3(q_3) \, {}^3\boldsymbol{T}_e \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix} = \begin{pmatrix} q_2 - L\sin q_3 \\ q_1 + L\cos q_3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} {}^w\boldsymbol{p}_e \\ 1 \end{pmatrix}.
$$

Figure 6 shows the primary workspace of the PPR planar robot for the given joint ranges. The darker yellow region shows the excursion of the two prismatic joints; the bottom-left corner is cut away by the length L of the third link and specifically by the lower limit $q_3 = -3\pi/4$ of the revolute joint (i.e., the last link points at most 45° downwards).

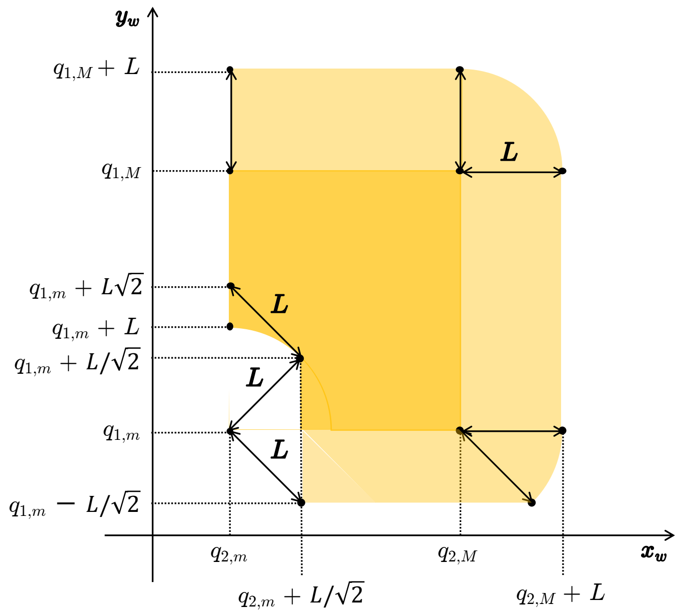

Figure 6: Primary workspace of the PPR robot, with the relevant points of interest.

Exercise 4

The transmission ratios of interest are

$$
n_{me} = \frac{r_e}{r_m} = \frac{40}{0.5} = 80 \qquad n_{el} = \frac{r_l}{r_e} = \frac{10}{40} = 0.25 \qquad \Rightarrow \qquad n = n_{me} \cdot n_{el} = \frac{r_l}{r_m} = 20.
$$

The matching condition for the link inertia that optimizes the transfer from motor torque to link acceleration is

$$
J_l = J_m n^2 = 7.1 \cdot 10^{-4}\, 400 = 0.284 \text{ kgm}^2.
$$

From the torque balance

$$
\tau_m = J_m \, \ddot{\theta}_m + \frac{1}{n} J_l \, \ddot{\theta}_l = J_m \left(n \, \ddot{\theta}_l \right) + \frac{1}{n} \left(J_m \, n^2 \right) \ddot{\theta}_l = 2 J_m \, n \, \ddot{\theta}_l,
$$

we compute the link acceleration

$$\ddot{\theta}_l = \frac{\tau_m}{2nJ_m} = \frac{10}{40 \cdot 7.1 \cdot 10^{-4}} = 352 \text{ rad/s}^2,$$

which is indeed a very large value (because the delivered torque τ_m is already extremely high!). To obtain the desired resolution Δ_l on the link side, we have on the encoder axis

$$\Delta_e = n_{el}\,\Delta_l = n_{el}\,0.01^\circ = 0.25\,\frac{0.01}{360} = \frac{2.5 \cdot 10^{-3}}{360} \text{ fraction of a turn.}$$

Thus, the number of pulses per turn of the incremental encoder should be

$$N = \frac{1}{\Delta_e} = \frac{360}{2.5 \cdot 10^{-3}} = 144000.$$

Accordingly, the pulses per turn of the optical disc are at least $N_e = \lceil N/4 \rceil = 36000$ (i.e., before electronic quadrature). When counting an increment of 100 pulses per second on the encoder axis, the motor velocity will be

$$\dot{\theta}_m = -n_{me}\,\dot{\theta}_e = -80\,\frac{100}{144000} = -0.056 \text{ turns/s } (\cdot 2\pi = -0.349 \text{ rad/s,})$$

with the sign $-$ due to the inverse rotation between motor and encoder (whereas $\text{sign}(\dot{\theta}_l) = \text{sign}(\dot{\theta}_m)$).

Exercise 5

Using the set of DH parameters given in Tab. 1, the unique corresponding assignment of DH frames is given in Fig. 7. In the shown configuration, we have $\boldsymbol{q} = (\pi/4, q_2 > 0, q_3 > 0, \pi/2)$.

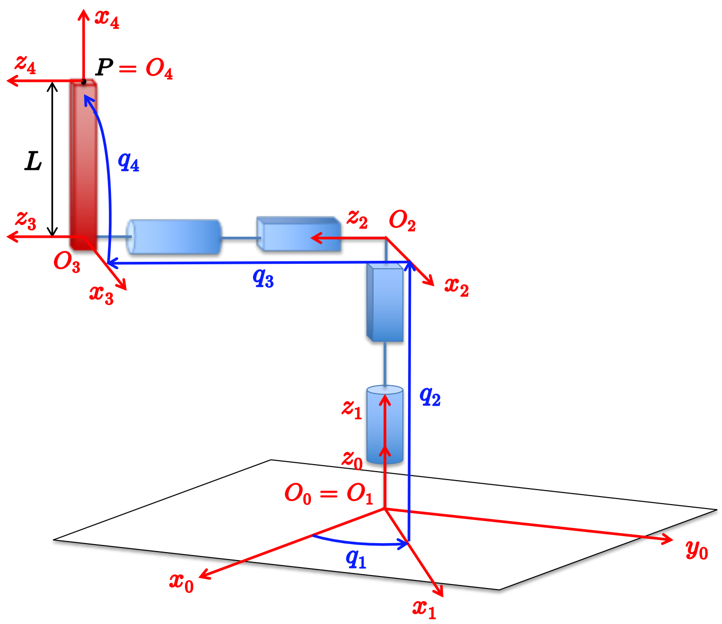

Figure 7: Assignment of DH frames for the RPPR robot of Fig. 4.

For completeness, the final pose of the frame RF_4 is expressed by the homogeneous transformation matrix

$${}^0\boldsymbol{T}_4(\boldsymbol{q}) = \begin{pmatrix} \cos q_1 \cos q_4 & -\cos q_1 \sin q_4 & \sin q_1 & \sin q_1 q_3 + L \cos q_1 \cos q_4 \\ \sin q_1 \cos q_4 & -\sin q_1 \sin q_4 & -\cos q_1 & -\cos q_1 q_3 + L \sin q_1 \cos q_4 \\ \sin q_4 & \cos q_4 & 0 & q_2 + L \sin q_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

One can easily recognize that the first three components of the task vector $\boldsymbol{r}$ in eq. (1) coincide with the position ${}^0\boldsymbol{p}_4$. For a given $\boldsymbol{r}_d = (p_{xd}, p_{yd}, p_{zd}, \alpha_d)$, the inverse kinematics (IK) problem is solved as follows. First set

$$q_4 = \alpha_d, \tag{2}$$

and from the third equation in (1)

$$q_2 = p_{zd} - L\sin\alpha_d. \tag{3}$$

Replace (2) in the first two equations in (1), which are rearranged as

$$\begin{aligned} p_{xd} &= q_3 \sin q_1 + L\cos\alpha_d \cos q_1 \\ p_{yd} &= -q_3 \cos q_1 + L\cos\alpha_d \sin q_1. \end{aligned} \tag{4}$$

Squaring and summing yields

$$p_{xd}^2 + p_{yd}^2 = q_3^2 + L^2\cos^2\alpha_d,$$

and thus

$$q_3^{\{+,-\}} = \pm\sqrt{p_{xd}^2 + p_{yd}^2 - L^2\cos^2\alpha_d}. \tag{5}$$

When the argument of the square root in (5) is negative, the result is imaginary and the desired positionis outside the reachable workspace of the robot.[1] On the other hand, when $q_3 = 0$, we have a singular configuration. For each solution in (5), we obtain from (4) a linear system in the two unknowns $(\cos q_1, \sin q_1)$

$$\begin{pmatrix} L\cos\alpha_d & q_3^{\{+,-\}} \\ -q_3^{\{+,-\}} & L\cos\alpha_d \end{pmatrix} \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix} = \begin{pmatrix} p_{xd} \\ p_{yd} \end{pmatrix},$$

which is solved by

$$q_1^{\{+,-\}} = \text{ATAN2}\left\{p_{yd}L\cos\alpha_d + p_{xd}\,q_3^{\{+,-\}}, p_{xd}L\cos\alpha_d - p_{yd}\,q_3^{\{+,-\}}\right\} \tag{6}$$

under the assumption that the determinant of the coefficient matrix

$$L^2\cos^2\alpha_d + (q_3^{\{+,-\}})^2 = p_{xd}^2 + p_{yd}^2 > 0.$$

When $p_{xd}^2 + p_{yd}^2 = 0$, then in order to have a real value for q_3 from (5), it must be $\alpha_d = q_3 = \pm\pi/2$. In this case, q_1 is undefined and we are again in a singularity. The four numerical cases to be solved summarize all possible cases for the IK problem (with units of $\boldsymbol{q}$ being [rad,m,m,rad]):

$$\boldsymbol{r}_{d1} = \begin{pmatrix} 2 \\ 2 \\ 4 \\ -\pi/4 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{q}^{\{+\}} = \begin{pmatrix} 1.9718 \\ 5.0607 \\ 2.6220 \\ -0.7854 \end{pmatrix} \quad \boldsymbol{q}^{\{-\}} = \begin{pmatrix} -0.4010 \\ 5.0607 \\ -2.6220 \\ -0.7854 \end{pmatrix} \quad \text{(two regular solutions)}$$

$$\boldsymbol{r}_{d2} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ \pi/2 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{q} = \begin{pmatrix} \text{undefined} \\ 1,5 \\ 0 \\ 1.5708 \end{pmatrix} \quad \text{(singularity with infinite solutions)}$$

$$\boldsymbol{r}_{d3} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \text{no solution for } \boldsymbol{q} \quad \text{(the task data are out of the reachable workspace)}$$

$$\boldsymbol{r}_{d4} = \begin{pmatrix} 0 \\ 1.5 \\ 4 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{q} = \begin{pmatrix} 1.5708 \\ 4 \\ 0 \\ 0 \end{pmatrix} \quad \text{(singularity with only one solution).}$$

[1] One can consider the reachable workspace of this robot as a collection of primary workspaces $WS_1(\alpha)$, parametrized by the angle α (the fourth component in $\boldsymbol{r}$). The primary workspace WS_1 will be the *union* of the sets $WS_1(\alpha)$ over all possible values of α. A point $\boldsymbol{p} \in \mathbb{R}^3$ belongs to the secondary workspace WS_2 if it belongs to $WS_1(\alpha)$ for all possible values of α (thus, to the intersection of all sets $WS_1(\alpha)$).

In the last case, the point $\boldsymbol{p} = (0, 1.5, 4) \in \mathbb{R}^3$ is on the boundary of $WS_1(0)$, i.e., the primary workspace obtained for $\alpha = 0$. This boundary is the surface of an infinite cylinder having its main axis coincident with z_0 and radius $R = L = 1.5$ m; the primary workspace is the (unlimited) part of $\mathbb{R}^3$ in the *outside* of this surface.

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