BASICS IN PROBABILITY

1. EVENTS, ALGEBRAS & σ-ALGEBRAS

Consider an abstract set Ω (which we call the SPACE OF ELEMENTARY EVENTS), and a collection $\mathcal{F}$ of subsets of Ω (which we call EVENTS) satisfying the following properties:

$$\boxed{\begin{array}{l} 1)\ \emptyset \in \mathcal{F} \\ 2)\ A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F} \\ 3)\ A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}. \end{array}}$$

Clearly 1) and 2) imply that also Ω and $\emptyset$ are in $\mathcal{F}$. By De Morgan's laws it follows also:

$$3')\ A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}.$$

EXAMPLE. $\Omega = \{$facce di un dado$\}$ $= \{\{1\}, \{2\}, \dots, \{6\}\}$. Two examples of possible $\mathcal{F}$ are

$$\mathcal{F}_1 = \{\emptyset, \{1\}, \dots, \{6\}, \{1,2\}, \dots, \{1,6\}, \dots, \{1,2,3,\dots,6\}\}$$

$$\mathcal{F}_2 = \{\emptyset, \{1,3,5\}, \{2,4,6\}, \{1,2,3,\dots,6\}\}$$

We call $\mathcal{F}$ the ALGEBRA OF EVENTS.

By finite induction from 3) it follows

$$\boxed{3'')\ A_i \in \mathcal{F},\ i=1,\dots,N \Rightarrow \bigcup_{i=1}^{N} A_i \in \mathcal{F},\quad N \in \mathbb{N}.}$$

When $N = +\infty$, $\mathcal{F}$ is a σ-ALGEBRA of EVENTS. Clearly, in this case, from 3'') we also have

$$\boxed{3''')\ A_i \in \mathcal{F},\ i=1,\dots,N \Rightarrow \bigcap_{i=1}^{N} A_i \in \mathcal{F},\quad N \in \mathbb{N}.}$$

The pair $(\Omega, \mathcal{F})$ is the (MEASURABLE) SPACE OF EVENTS.

<u>DEFINITION</u>. Given σ-algebras $\mathcal{F}_1, \mathcal{F}_2$, $\mathcal{F}_1$ is less rich than $\mathcal{F}_2$ (we write $\mathcal{F}_1 \subseteq \mathcal{F}_2$) if

$$\forall A \in \mathcal{F}_1 \Rightarrow A \in \mathcal{F}_2$$

DEFINITION An ATOM of a σ-algebra $\mathcal{F}$ is an event which cannot be obtained as union of other events

In conclusion, a σ-algebra $\mathcal{F}$ on Ω satisfies:

1) $\emptyset \in \mathcal{F}$
2) $A^c \in \mathcal{F}, \quad \forall A \in \mathcal{F}$
3) $A_i \in \mathcal{F}, \; i = 1, 2, \dots \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

In order to have a "measure" of the events (i.e. the elements of $\mathcal{F}$) we introduce a PROBABILITY MEASURE P on $\mathcal{F}$ which is a function

$P : \mathcal{F} \longrightarrow [0, 1]$ satisfying:

1) $P(\Omega) = 1$
2) $0 \leq P(A) \leq 1 \quad \forall A \in \mathcal{F}$
3) $A_i \in \mathcal{F}, \; i = 1, 2, \dots, \; A_i \cap A_j = \emptyset \quad \forall i, j \Rightarrow$
$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$
(COUNTABLE ADDITIVITY)

Since $\forall A \in \mathcal{F}: A \cup A^c = \Omega$
and $A \cap A^c = \phi$ and $\Omega, \phi \in \mathcal{F}$:

$$1 = \mathcal{P}(A \cup A^c) = \mathcal{P}(A) + \mathcal{P}(A^c)$$

$$\Rightarrow \boxed{\mathcal{P}(A^c) = 1 - \mathcal{P}(A)}$$

Ω is called the SURE EVENT, while
ϕ is called the IMPOSSIBLE EVENT.

REMARK. $\mathcal{P}(\phi) = 0$ since $\mathcal{P}(\Omega) = 1$
but $\mathcal{P}(A) = 0 \not\Rightarrow A = \phi$!. For instance,
if $\Omega = \{1, 2, 3, \dots\}$ and $\mathcal{P}(\{i\}) = \mathcal{P}(\{j\})$
then $\mathcal{P}(\{i\}) = 0$ since $\mathcal{P}(\Omega) = 1$.

Two simple examples of σ-algebras on Ω
are

$$\mathcal{F}_M = \{\text{set of all subset of } \Omega\}$$

and

$$\mathcal{F}_m = \{\phi, \Omega\}$$

Clearly

$$\boxed{\mathcal{F}_m \subseteq \mathcal{F} \subseteq \mathcal{F}_M}$$

for any σ-algebra $\mathcal{F}$ on Ω.

Dato un insieme $\mathcal{C}$ of subsets of Ω we want to characterize a σ-algebra which contains $\mathcal{C}$. Clearly, $\mathcal{F}_M$ is one such σ-algebra. Let $\mathcal{F}_1, \mathcal{F}_2$ be two σ-algebras containing $\mathcal{C}$. $\mathcal{F}_1 \cap \mathcal{F}_2$ is itself a σ-algebra and contains $\mathcal{C}$: indeed,

- $\emptyset \in \mathcal{F}_1 \cap \mathcal{F}_2$ (since $\emptyset \in \mathcal{F}_1, \mathcal{F}_2$)
- $\Omega \in \mathcal{F}_1 \cap \mathcal{F}_2$ (since $\Omega \in \mathcal{F}_1, \mathcal{F}_2$)
- $A_i \in \mathcal{F}_1 \cap \mathcal{F}_2,\ i = 1, 2, \ldots \Rightarrow A_i \in \mathcal{F}_1, \mathcal{F}_2 \Rightarrow$
  $\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_1, \mathcal{F}_2 \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_1 \cap \mathcal{F}_2$
- $A \in \mathcal{F}_1 \cap \mathcal{F}_2 \Rightarrow A \in \mathcal{F}_1, \mathcal{F}_2 \Rightarrow A^c \in \mathcal{F}_1, \mathcal{F}_2$
  $\Rightarrow A^c \in \mathcal{F}_1 \cap \mathcal{F}_2$

Therefore, if $\{\mathcal{F}_\alpha\}$ is the family of σ-algebras containing $\mathcal{C}$

$$\mathcal{F} = \bigcap_{\alpha} \mathcal{F}_\alpha$$

is a σ-algebra containing $\mathcal{C}$ and it is the smallest with this property. We call $\mathcal{F}$ the σ-algebra GENERATED BY $\mathcal{C}$ and we denote it by $\sigma(\mathcal{C})$.

EXAMPLE. If $\Omega = [0,1]$ and $\mathcal{C} = \{[0,\frac{1}{3}], [\frac{2}{3},1]\}$ then

$$\sigma(\mathcal{C}) = \{[0,\tfrac{1}{3}], [\tfrac{2}{3},1], \emptyset, (\tfrac{1}{3},1], [0,\tfrac{2}{3}), [0,1], [0,\tfrac{1}{3}] \cup [\tfrac{2}{3},1], (\tfrac{1}{3},\tfrac{2}{3})\}$$ ◀

If $\Omega = \mathbb{R}$, it is common practice to consider on Ω the BOREL σ-algebra $\mathcal{B}(\mathbb{R})$ which is the σ-algebra generated by $\mathcal{C} = \{$the set of open intervals of $\mathbb{R}\} = \{(a,b) : a, b \in \mathbb{R}\}$.
It can be shown that $\mathcal{B}(\mathbb{R})$ can be generated by different sets $\mathcal{C}$:

$$\mathcal{C}_1 = \{(a,+\infty), a \in \mathbb{R}\}$$

$$\mathcal{C}_2 = \{(-\infty,a), a \in \mathbb{R}\}$$

$$\mathcal{C}_3 = \{[a,b], a,b \in \mathbb{R}\}$$

$$\mathcal{C}_4 = \{[a,b), a,b \in \mathbb{R}\}$$

Similarly, we can generate the σ-algebra $\mathcal{B}(\mathbb{R}^n)$ (with $\Omega = \mathbb{R}^n$).

2. RANDOM VARIABLES

DEFINITION. A RANDOM VARIABLE on $(\Omega, \mathcal{F})$ is a function $X : \Omega \to \mathbb{R}$ such that

$$\boxed{X^{-1}(B) \triangleq \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F} \quad \forall\, B \in \mathcal{B}(\mathbb{R})}$$

REMARK. We say that X is $\mathcal{F}$-measurable.

In $\mathcal{F}_m$ the random variables X are constant, while in $\mathcal{F}_M$ any function $X : \Omega \to \mathbb{R}$ is a random variable (since $\mathcal{F} \subseteq \mathcal{F}_M$ for any σ-algebra $\mathcal{F}$ on Ω).

If X is a random variable in $\mathcal{F}$ it is a random variable in any other $\mathcal{F}' \supset \mathcal{F}$

DEFINITION. The smallest σ-algebra $\mathcal{F}^X$ for which a random variable X on $(\Omega, \mathcal{F})$ is $\mathcal{F}^X$-measurable is the σ-algebra GENERATED BY X.

We have

$$\boxed{\mathcal{F}^X \triangleq \{X^{-1}(B),\ B \in \mathcal{B}(\mathbb{R})\}}$$

$\mathcal{F}^X$ is indeed a σ-algebra:

1) $B = \emptyset \Rightarrow X^{-1}(B) = \emptyset \Rightarrow \emptyset \in \mathcal{F}^X$

2) $A \in \mathcal{F}^X \Rightarrow \exists\ B \in \mathcal{B}(\mathbb{R}^n) : A = X^{-1}(B)$

$\Rightarrow B^c \in \mathcal{B}(\mathbb{R}^n) \Rightarrow X^{-1}(B^c) = A^c \in \mathcal{F}^X$

3) $A_i \in \mathcal{F}^X,\ i = 1, 2, \ldots \Rightarrow \exists\ B_i \in \mathcal{B}(\mathbb{R}^n), i = 1, 2, \ldots :$

$A_i = X^{-1}(B_i) \Rightarrow \bigcup_i B_i \in \mathcal{B}(\mathbb{R}^n)$

$\Rightarrow X^{-1}(\bigcup_i B_i) \in \mathcal{F}^X \Rightarrow \bigcup_i A_i = \bigcup_i X^{-1}(B_i)$

$= X^{-1}(\bigcup_i B_i) \in \mathcal{F}^X.$

<u>EXAMPLE</u> Given $(\Omega, \mathcal{F}, P)$, the function

$$\chi_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

where $A \in \mathcal{F}$, is a random variable on $(\Omega, \mathcal{F})$. χ_A is called CHARACTERISTIC FUNCTION OF A.

REMARK. To check out if X is a random variable it is sufficient to check the definition with $\mathcal{B}$ any element of the set of generators for $\mathcal{B}(\mathbb{R})$. For example,

$$\mathcal{B} = \{(-\infty, a] ; a \in \mathbb{R}\}$$

Therefore,

$$\chi_A^{-1}((-\infty, a]) = \phi \in \mathcal{F} \quad \text{if } a<0$$

$$\chi_A^{-1}((-\infty, a]) = A^c \in \mathcal{F} \quad \text{if } 0 \leq a < 1$$

$$\chi_A^{-1}((-\infty, a]) = A \cup A^c = \Omega \in \mathcal{F} \quad \text{if } a \geq 1$$

Moreover,

$$\mathcal{F}^{X_A} = \{\phi, A, A^c, \Omega\}$$

with $\mathcal{F}^{X_A} \subseteq \mathcal{F}$.

PROPERTIES OF $\chi_A(\omega)$.

1) $c \in \mathbb{R} \Rightarrow c\chi_A$ is a random variable.
2) $A_1, A_2 \in \mathcal{F}$, $c_1, c_2 \in \mathbb{R} \Rightarrow c_1\chi_{A_1} + c_2\chi_{A_2}$ is a random variable.
3) $A_i \in \mathcal{F}$, $c_i \in \mathbb{R}$, $i=1,2,\dots,N \Rightarrow \sum_{i=1}^{N} c_i \chi_{A_i}$ is a random variable.

3. EXPECTATION OF RANDOM VARIABLES

DEFINITION A function ψ is said to be SIMPLE if

$$\psi(\omega) = \sum_{i=1}^{N} c_i \chi_{A_i}(\omega)$$

with $c_i \in \mathbb{R}$, $\bigcup_{i=1}^{N} A_i = \Omega$

The sets A_i are not disjoint, but we can always assume that A_i are disjoint by re-writing ψ. Moreover, ψ is a random variable.

We define the integral of a simple nonnegative random variable ψ as follows:

$$\boxed{\int_\Omega \psi(\omega)\, d\mathcal{P}(\omega) \triangleq \sum_{i=1}^{N} c_i \mathcal{P}(A_i)}$$

Next step is to consider any nonnegative random variable $X(\omega)$ and approximate it as a sequence of simple nonnegative functions $\psi^n(\omega)$.

Let

$$A_i \triangleq X^{-1}\left(\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right)\right)$$

$$= \left\{\omega : \frac{i}{2^n} \leq X(\omega) < \frac{i+1}{2^n}\right\}$$

$$C_n \triangleq \{\omega : X(\omega) \geq n\}$$

If

$$\boxed{\psi^n(\omega) \triangleq \sum_{i=0}^{n2^n-1} \frac{i}{2^n} \chi_{A_i}(\omega) + n\chi_{C_n}(\omega)}$$

then $\forall\, \omega \in \Omega$

$$\boxed{\lim_{n\to+\infty} \psi^n(\omega) = X(\omega)}$$

(notice $\psi^n(\omega) \leq \psi^{n+1}(\omega) \leq X(\omega)$
$\forall\, n$ and for each $\omega \in \Omega$).

Since ψ^n is nonnegative and simple:

$$\boxed{\begin{aligned} I^n &\triangleq \int_\Omega \psi^n(\omega)\, d\mathcal{P}(\omega) \\ &= \sum_{i=0}^{n2^n-1} \frac{i}{2^n} \mathcal{P}(A_i) + n\mathcal{P}(C_n) \end{aligned}}$$

(notice that $I^n \leq I^{n+1}$ $\forall\, n$)

On account of the previous fact we can define the integral of a nonnegative random variable X on Ω as:

$$\boxed{\int_{\Omega} X(\omega)\, dP(\omega) \triangleq \underbrace{\lim_{n\to+\infty} \int \psi^n(\omega)\, dP(\omega)}_{\displaystyle \lim_{n\to+\infty} I^n}}$$

whenever the limit on the right exists. In this case, we say that X is INTEGRABLE. It is easy to extend the above definition to any random variable $X(\omega)$. Write

$$X(\omega) = X^+(\omega) - X^-(\omega)$$

where $X^+(\omega) = \begin{cases} 0 & \text{if } X(\omega) \leq 0 \\ X(\omega) & \text{if } X(\omega) > 0 \end{cases}$

$$X^-(\omega) = \begin{cases} 0 & \text{if } X(\omega) \geq 0 \\ -X(\omega) & \text{if } X(\omega) < 0 \end{cases}$$

and define

$$\int_{\Omega} X(\omega)\, dP(\omega) = \int_{\Omega} X^+(\omega)\, dP(\omega) - \int_{\Omega} X^-(\omega)\, dP(\omega)$$

EXAMPLE. In a small town there are 20 coffee-shops, divided into 3 groups according to the prices

1.4 € in group A (4 coffeeshops)
1.2 € in group B (10 coffee shops)
1.0 € in group C (6 coffee shops)

What is the expected price in this town?

- Ω = { insieme dei coffee shops }
- $\mathcal{F}_m$ is the σ-algebra on Ω
- $\mathcal{P}$ acts on the elements of $\mathcal{F}_m$ by giving the ratio between the cardinality of the element of $\mathcal{F}_m$ and the cardinality of Ω

The function PRICE is:

$$\psi(\omega) = 1{,}4\,\chi_A(\omega) + 1.2\,\chi_B(\omega) + 1.0\,\chi_C(\omega)$$

We can calculate

$$\int_\Omega \psi(\omega)\, d\mathcal{P}(\omega) = 1{,}4\,\mathcal{P}(A) + 1{,}2\,\mathcal{P}(B) + 1{,}0\,\mathcal{P}(C)$$

but

$$\mathcal{P}(A) = \frac{4}{20}$$

$$\mathcal{P}(B) = \frac{10}{20}$$

$$\mathcal{P}(C) = \frac{6}{20} \quad \Rightarrow$$

$$\int_\Omega \psi(\omega)\, d\mathcal{P}(\omega) = 1{,}18 \text{ €}$$

We call

$$\boxed{\int_\Omega X(\omega)\, d\mathcal{P}(\omega)}$$

the EXPECTATION of $X(\omega)$.

The expectation of X is well-defined if the expectation of either X^+ or X^- is well-defined. If $E\{X^\pm\} < +\infty$ then X is INTEGRABLE.

DEFINITION The VARIANCE of a random variable $X : \Omega \to \mathbb{R}$ (or centered second-order moment) is

$$\boxed{\sigma_X^2 \triangleq E\{(X - E\{X\})^2\}}$$

σ_X is the STANDARD DEVIATION.

EXAMPLE (continued)

The variance of the prices in the coffee-shops of the town is

$$\sigma_\psi^2 = \frac{4}{20} \cdot (0.22)^2 + \frac{10}{20} (0.02)^2 + \frac{6}{20} (0.18)^2$$

$$= 0.0196 \; €^2$$

$$\sigma_\psi \cong 0.140 \; €$$

Notice that if the prices would be different as:

1.1 € in group A
1.5 € in group B
0.7 € in group C

then

$$E\{\psi(\omega)\} = 1,18 \text{ €}$$

(the same expected price as before!)

but

$$\sigma_\psi^2 = 0.1216 \text{ €}^2$$

$$\sigma_\psi = 0,3487 \text{ €}$$

4. DISTRIBUTION AND DENSITY OF RANDOM VARIABLES

DEFINITION. Given $(\Omega, \mathcal{F}, P)$, the distribution function $F_X : \mathbb{R} \to [0,1]$ of a random variable X is defined as

$$\boxed{F_X(x) = P\{\omega \in \Omega : X(\omega) \in (-\infty, x)\}}$$

Main properties of $F_X(x)$:

1) $\lim_{x \to +\infty} F_X(x) = 1$

2) $\lim_{x \to -\infty} F_X(x) = 0$

3) $F_X(x)$ is monotonically non decreasing:

$$\boxed{F_X(x_1) \geq F_X(x_2) \text{ for } x_1 \geq x_2}$$

4) $F_X(x)$ is continuos from the left:

$$\boxed{\lim_{x \to x_0^-} F_X(x) = F_X(x_0), \ \forall x_0 \in R}$$

REMARK.

$$\boxed{\begin{aligned} &\mathcal{P}(\{\omega \in \Omega : x_1 \leq X(\omega) < x_2\}) = F_X(x_2) - F_X(x_1) \\ &\mathcal{P}(\{\omega \in \Omega : X(\omega) \geq x\}) = 1 - F_X(x) \end{aligned}}$$

The distribution function $F_X(x)$ can be used to calculate the expectation of X.

With reference to pg. 11, for a nonnegative X

$$\lim_{n \to +\infty} \psi^n(\omega) = X(\omega)$$

where

$$\psi^n(\omega) \triangleq \sum_{i=0}^{n2^n - 1} \frac{i}{2^n} \chi_{A_i}(\omega) + n \chi_{C_n}(\omega)$$

and

$$A_i \triangleq X^{-1}\left(\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right)\right)$$

$$= \left\{\omega : \frac{i}{2^n} \leq X(\omega) < \frac{i+1}{2^n}\right\}$$

$$C_n = \{\omega : X(\omega) \geq n\}.$$

using F_X we have

$$A_i = F_X\left(\frac{i+1}{2^n}\right) - F_X\left(\frac{i}{2^n}\right)$$

$$C_n = 1 - F_X(n)$$

then

$$E[\psi^n(\omega)] = \sum_{i=0}^{n2^n-1} \frac{i}{2^n}\left[F_X\left(\frac{i+1}{2^n}\right) - F_X\left(\frac{i}{2^n}\right)\right] + n\left[1 - F_X(n)\right]$$

and

$$E[X(\omega)] = \lim_{n\to\infty}\left\{\sum_{i=0}^{n2^n-1} \frac{i}{2^n}\left[F_X\left(\frac{i+1}{2^n}\right) - F_X\left(\frac{i}{2^n}\right)\right] + \boxed{n\left[1 - F_X(n)\right]}\right\}$$

$$\triangleq \int_0^\infty x\, dF_X(x)$$

$\boxed{n[1-F_X(n)] \to 0 \text{ as } n\to\infty}$

(whenever the limit exists)

which is a Stilties integral.

If F_X is absolutely continuous (in particular, almost everywhere differentiable):

$$E[X(\omega)] = \lim_{n\to+\infty} \left\{ \sum_{i=0}^{n2^n-1} \frac{i}{2^n} \left[F_X\left(\frac{i+1}{2^n}\right) - F_X\left(\frac{i}{2^n}\right) \right] + n\left[1 - F_X(n)\right] \right\}$$

$$= \lim_{n\to+\infty} \sum_{i=0}^{n2^n-1} \frac{i}{2^n} \frac{F_X\left(\frac{i+1}{2^n}\right) - F_X\left(\frac{i}{2^n}\right)}{\frac{1}{2^n}} \frac{1}{2^n}$$

$$\triangleq \int_0^\infty x \frac{dF_X(x)}{dx} dx$$

which is a Lebesgue integral.

DEFINITION. The probability density function p_X of a random variable X is defined as

$$\boxed{p_X(x) = \frac{dF_X(x)}{dx}}$$

whenever F_X is absolutely continuous.

Therefore, for a random variable X

$$\boxed{E\{X(\omega)\} = \int_{-\infty}^{\infty} x\, p_X(x)\, dx}$$

Notice that

$$\int_{-\infty}^{\infty} dF_X(x) = F_X(+\infty) - F_X(-\infty)$$

$$= 1$$

and if F_X is differentiable:

$$\boxed{\int_{-\infty}^{\infty} \frac{dF_X(x)}{dx} \cdot dx = \int_{-\infty}^{\infty} p_X(x)\, dx = 1}$$

By summing up, the expectation $E\{X(\omega)\}$ of a random variable X can be calculated in three ways:

1) $E\{X(\omega)\} = \int_{\Omega} X(\omega)\, d\mathcal{P}(\omega)$

using the probability measure $\mathcal{P}$,

2) $E\{X(\omega)\} = \int_{-\infty}^{\infty} x\, dF_X(x)$

using the distribution function F_X,

3) $E\{X(\omega)\} = \int_{-\infty}^{\infty} x\, p_X(x)\, dx$

using the density p_X.

<u>EXAMPLE</u>. $X(\omega) = c \quad \forall\, \omega$,

with $c \in \mathbb{R}$; X is a random variable and it is simple:

$$X(\omega) = c = c\,\chi_{\Omega}(\omega)$$

Therefore

$$E\{X(\omega)\} = P(\Omega)\cdot c = c$$

But

$$F_X(x) = \begin{cases} 0 & \text{if} \quad x \leq c \\ 1 & \text{if} \quad x > c \end{cases}$$

and

$$E\{X(\omega)\} = \int_{-\infty}^{\infty} c \, dF_X(x) = c$$

The derivative of F_X is defined everywhere but in $x = c$. Moreover, it is not continuous at $x = c$.

For all $x \neq c$:

$$p_X(x) = \frac{d}{dx} F_X(x) = 0$$

and

$$\int_{-\infty}^{\infty} c \, \frac{dF_X(x)}{dx} dx = 0 \neq E\{X(\omega)\}!$$

5. INDEPENDENT RANDOM VARIABLES

DEFINITION. Given $(\Omega, \mathcal{F}, P)$ a group of events $A_1, \dots, A_N \in \mathcal{F}$ are said INDEPENDENT if

$$P\left(\bigcap_{i=1}^{N} A_i\right) = \prod_{i=1}^{N} P(A_i)$$

EXAMPLE. $\Omega = \{\omega_1, \dots, \omega_4\}$ and the σ-algebra $\mathcal{F}_M$ on Ω. Moreover, P is defined in such a way that $P(\{\omega_i\}) = \frac{1}{4}$, $i = 1, \dots, 4$. Consider

$$A_1 = \{\omega_1, \omega_2\}$$

$$A_2 = \{\omega_2, \omega_4\}$$

$$A_3 = \{\omega_1, \omega_4\}$$

Clearly, $P(A_i) = \frac{1}{2}$ and

$$P(A_i \cap A_j) = \frac{1}{4} = P(A_i) P(A_j)$$

for $i \neq j$

which means that the pairs of events (A_i, A_j), $i \neq j$, are independent. But

$$P(A_1 \cap A_2 \cap A_3) = \frac{1}{4} \neq \frac{1}{8}$$
$$= P(A_1)P(A_2)P(A_3)$$

EXAMPLE. Consider n tosses of a coin with $P(\text{head}) = P(\text{cross}) = \frac{1}{2}$. In the overall, we have 2^n possible sequences. The possible sequences with k crosses are $\binom{n}{k}$ and the probability of having k crosses is $\binom{n}{k}\frac{1}{2^n}$. If

$A = \{$ sequences of tosses with at least one head and cross $\}$

$B = \{$ sequences of tosses with at most one cross $\}$

then

$$P(A) = 1 - \frac{1}{2^n} \cdot \boxed{2} = \frac{2^{n-1} - 1}{2^{n-1}}$$

sequences with either all crosses or heads

$$P(B) = \frac{\boxed{n+1}}{2^n}$$

sequences with at most one cross

and

sequences with exactly one cross

$$P(A \cap B) = \frac{\boxed{n}}{2^n}$$

so that

$$P(A \cap B) = P(A) \cdot P(B)$$

if $n = 3$

but

$$P(A \cap B) \neq P(A) \cdot P(B)$$

if $n = 2$

Independence is a mathematical notion !

(not physical)

DEFINITION. Two σ-algebras $\mathcal{F}_1, \mathcal{F}_2$ on Ω are said INDEPENDENT ($\mathcal{F}_1 \perp \mathcal{F}_2$) if $\forall A_1 \in \mathcal{F}_1$, $\forall A_2 \in \mathcal{F}_2$, A_1 and A_2 are independent.

A σ-algebra is said to be ATOMIC if it is generated by a countable set of disjoint sets (ATOMS). If $\{A_i\}_{i \in \mathbb{N}}$ is this family, we will write $\sigma\{A_i\}$ for the atomic σ-algebra.
For two atomic σ-algebras $\sigma\{A_i\}$ and $\sigma\{B_i\}$ independency can be checked out on the atoms:

$$P(A_i \cap B_j) = P(A_i)\,P(B_j)$$
$$\forall i,j \in \mathbb{N}$$

As a matter of fact, for an atomic σ-algebra any event A can be written as the union of atoms A_i: if $A = A_i \cup A_h$ and since A_i and A_h are disjoint

$$\mathcal{P}(A \cap B_j') = \mathcal{P}((A_i' \cup A_h') \cap B_j')$$
$$= \mathcal{P}((A_i' \cap B_j') \cup (A_h' \cap B_j'))$$
$$= \mathcal{P}(A_i' \cap B_j') + \mathcal{P}(A_h' \cap B_j')$$
$$= (\mathcal{P}(A_i') + \mathcal{P}(A_j'))\mathcal{P}(B_j') = \mathcal{P}(A)\mathcal{P}(B_j')$$

if the atoms of $\sigma\{A_i'\}$ and $\sigma\{B_j'\}$ are independent!

independence of A and B_j'!

DEFINITION. Two random variables $X_1(\omega)$ and $X_2(\omega)$ on $(\Omega, \mathcal{F}, \mathcal{P})$ are said INDEPENDENT if the σ-algebras $\mathcal{F}^{X_1}$ and $\mathcal{F}^{X_2}$ are independent. We will write

$X_1 \perp X_2$

For two simple independent random variables ψ_1, ψ_2:

$$E\{\psi_1 \psi_2\} = E\{\psi_1\} E\{\psi_2\}$$

As a matter of fact,

if $A \in \mathcal{F}^{\psi_1}$ and $B \in \mathcal{F}^{\psi_2}$

$$E\{\chi_A \chi_B\} = \int_\Omega \chi_A \chi_B \, dP$$

$$= \int_{A \cap B} dP = P(A \cap B) = P(A) P(B)$$

$$= \int_A dP \int_B dP = \int_\Omega \chi_A \, dP \int_\Omega \chi_B \, dP$$

$$= E\{\chi_A\} E\{\chi_B\}$$

Since ψ_1, ψ_2 are simple:

$$\psi_1 = \sum_{i=1}^{n} \alpha_i \chi_{A_i}, \quad \psi_2 = \sum_{j=1}^{m} \beta_j \chi_{B_j}$$

we can write

$$\psi_1 = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \chi_{A_i \cap B_j}, \quad \psi_2 = \sum_{i=1}^{n} \sum_{j=1}^{m} \beta_j \chi_{A_i \cap B_j}$$

therefore

$$\psi_1 \psi_2 = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j \chi_{A_i \cap B_j}$$

and

$$E\{\psi_1\psi_2\} = \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i \beta_j P(A_i \cap B_j)$$

$$= \sum_{i=1}^{n} \alpha_i P(A_i) \sum_{j=1}^{m} \beta_j P(B_j)$$

$$= E\{\psi_1\} E\{\psi_2\}$$

By straightforward extension to independent random variables X_1, X_2:

$$\boxed{E\{X_1 X_2\} = E\{X_1\} E\{X_2\}}$$

and, more generally,

$$\boxed{E\{f(X_1) g(X_2)\} = E\{f(X_1)\} E\{g(X_2)\}}$$

for any measurable functions $f(\cdot), g(\cdot)$ (as a matter of fact $f(X_1)$ and $g(X_2)$ are $\mathcal{F}^{X_1}$ and, respectively, $\mathcal{F}^{X_2}$ measurable)

DEFINITION. For two random variable X, Y, the covariance of X and Y, denoted by σ_{XY}, is:

$$\sigma_{XY} \triangleq E\{(X-E\{X\})(Y-E\{Y\})\}$$

We have

$$\sigma_{XY} = \underbrace{E\{XY\}}_{\substack{\text{this is known} \\ \text{as CORRELATION} \\ \text{between } X \text{ and } Y}} - E\{X\}E\{Y\}$$

FACT. If $X_1 \perp X_2$ are random variables with $\sigma^2_{X_1}$ and, respectively, $\sigma^2_{X_2}$:

$$\sigma^2_{X_1+X_2} = \sigma^2_{X_1} + \sigma^2_{X_2}$$

Indeed, since $X_1 \perp X_2$

$$\sigma_{X_1X_2} = E\{(X_1-E\{X_1\})(X_2-E\{X_2\})\}$$
$$= E\{X_1-E\{X_1\}\}E\{X_2-E\{X_2\}\} = 0$$

Moreover,

$$\sigma^2_{X_1+X_2} = E\{(X_1+X_2-E\{X_1+X_2\})^2\}$$

$$= E\{(X_1-E\{X_1\}+X_2-E\{X_2\})^2\}$$

$$= E\left\{\sum_{i=1}^{2}\sum_{j=1}^{2}(X_i-E\{X_i\})(X_j-E\{X_j\})\right\}$$

$$= \sum_{i=1}^{2}\sum_{j=1}^{2}E\{(X_i-E\{X_i\})(X_j-E\{X_j\})\}$$

$$= \sum_{i=1}^{2}E\{(X_i-E\{X_i\})^2\} = \sigma^2_{X_1}+\sigma^2_{X_2}$$

For two random variables X, Y:

<u>DEFINITION</u> CORRELATION COEFFICIENT of X and Y (PEARSON)

$$\triangleq \frac{\sigma_{XY}}{\sigma_X\sigma_Y} \triangleq \rho_{X_1X_2}$$

Notice $\rho_{X_1X_2} \in [-1,1]$ since

$$E\{X^2\}E\{Y^2\} - E^2\{XY\} \geq 0$$

(from $E\{(XE\{Y^2\}-YE\{XY\})^2\} \geq 0$)

Two random variables X_1, X_2 are said UNCORRELATED if

$$\rho_{X_1 X_2} = 0$$

For uncorrelated X_1 and X_2 we have also:

$$\begin{cases} E\{X_1 X_2\} = E\{X_1\} E\{X_2\} \\ \sigma^2_{X_1+X_2} = \sigma^2_{X_1} + \sigma^2_{X_2} \end{cases}$$

However,

INDEPENDENCE $\Rightarrow$ UNCORRELATION

UNCORRELATION $\not\Rightarrow$ INDEPENDENCE

<u>EXAMPLE</u>. $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, $\mathcal{F} = \mathcal{F}_M$

and
$$P(\omega_1) = \frac{1}{9}, \; P(\omega_2) = \frac{2}{9}, \; P(\omega_3) = \frac{2}{9}, \; P(\omega_4) = \frac{4}{9}$$

The random variables

$$X_1(\omega) = \begin{cases} 1 & \omega \in \{\omega_1, \omega_2\} \\ 3 & \omega \in \{\omega_3, \omega_4\} \end{cases} \qquad X_2(\omega) = \begin{cases} 1 & \omega \in \{\omega_1, \omega_3\} \\ 2 & \omega \in \{\omega_2, \omega_4\} \end{cases}$$

are independent.

Indeed

$$\mathcal{F}^{X_1} = \{\phi, A, A^c, \Omega\}$$

$$A \triangleq \{\omega_1, \omega_2\}$$

$$A^c = \{\omega_3, \omega_4\}$$

and

$$\mathcal{F}^{X_2} = \{\phi, B, B^c, \Omega\}$$

$$B \triangleq \{\omega_1, \omega_3\}, \quad B^c = \{\omega_2, \omega_4\}$$

with

$$\mathcal{P}(A) = \frac{1}{3}, \mathcal{P}(A^c) = \frac{2}{3}, \mathcal{P}(B) = \frac{1}{3}, \mathcal{P}(B^c) = \frac{2}{3}$$

so that

$$\mathcal{P}(A \cap B) = \mathcal{P}(\omega_1) = \frac{1}{9}, \mathcal{P}(A^c \cap B) = \mathcal{P}(\omega_3) = \frac{2}{9}$$

$$\mathcal{P}(A \cap B^c) = \mathcal{P}(\omega_2) = \frac{2}{9}, \mathcal{P}(A^c \cap B^c) = \mathcal{P}(\omega_4) = \frac{4}{9}$$

EXAMPLE. (X, Y) be random variables assuming $(-1, 1)$, $(0, 0)$, $(1, 1)$ with probability $1/4$, $1/2$, $1/4$, respectively.

$$E\{Y\} = E\{X\} = -1 \cdot 1/4 + 0 \cdot 1/2 + 1 \cdot 1/4 = 0$$

$$E\{XY\} = -1 \cdot 1/4 + 0 \cdot 1/2 + 1 \cdot 1/4 = 0$$

$\Rightarrow X, Y$ are uncorrelated! But

$$\mathcal{P}\{X = -1, Y = 1\} = \frac{1}{4} \neq \frac{1}{16} = \mathcal{P}\{X = -1\}\mathcal{P}\{Y = 1\}$$

$\Rightarrow X, Y$ are NOT independent!

6. SOME DISTRIBUTIONS

6.1 BINOMIAL DISTRIBUTION

Consider N random variables on $(\Omega, \mathcal{F}, \mathcal{P})$ X_i, $i=1,\dots,N$, independent and such that

$$\boxed{\begin{aligned} &\mathcal{P}\{\omega : X_i(\omega)=1\}=p \\ &\mathcal{P}\{\omega : X_i(\omega)=0\}=1-p \end{aligned}}$$

Let determine the distribution of

$$X(\omega)=\sum_{i=1}^{N} X_i(\omega)$$

The number of possible sequences of N zeroes and ones with sum equal to n is $\binom{N}{n}$. Each sequence has probability $p^n(1-p)^{N-n}$, therefore

$$\mathcal{P}\{X(\omega)=n\}=\binom{N}{n}p^n(1-p)^{N-n}$$

The distribution function $F_X(n)$ is

$$F_X(n)=\mathcal{P}\{X(\omega)\leq n\}=\sum_{k=0}^{n}\binom{N}{k}p^k(1-p)^{N-k}$$

Notice that

$$1 = (p+1-p)^N = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k}$$

$$= F_X(N)$$

Finally, using the distribution function,

$$E\{X(\omega)\} = \sum_{n=0}^{N} n \binom{N}{n} p^n (1-p)^{N-n} = Np$$

$$\sigma_X^2 = \sum_{n=0}^{N} (n-Np)^2 \binom{N}{n} p^n (1-p)^{N-n} = Npq$$

6.2 POISSON DISTRIBUTION

If $N \to \infty$ and $p \to 0$ in such a way that

$$Np = a$$

with given a, we obtain the POISSON distribution. With $n \ll N$

$$\frac{N!}{(N-n)!} = N(N-1)\dots(N-n+1) \simeq N^n$$

and from the binomial distribution

$$\mathcal{P}\{X(\omega)=n\} = \binom{N}{n} p^n (1-p)^{N-n}$$

$$\approx \frac{(Np)^n}{n!} (1-p)^{N-n}$$

$$\approx \frac{a^n}{n!} (1-p)^{\frac{a}{p}} \qquad (\text{since } n \ll N)$$

and for $p \to 0$:

$$\boxed{\mathcal{P}\{X(\omega)=n\} \approx \frac{a^n}{n!} e^{-a}}$$

with

$$F_X(n) = \sum_{k=0}^{n} \frac{a^k e^{-a}}{k!}$$

Notice $F_X(\infty) = 1$. Moreover,

$$E\{X(\omega)\} = a$$

$$\sigma_X^2 = a$$

6.3 GAUSSIAN DISTRIBUTION

We use the following integrals:

$$I \triangleq \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} .$$ This follows from:

$$I^2 \triangleq \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy =$$

$$\int_{-\infty}^{\infty} \int_{0}^{2\pi} e^{-\rho^2} \rho \, d\vartheta \, d\rho = \pi$$

also define

$$g(x, \sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

DEFINITION. A gaussian random variable $X(\omega)$ is such that

$$p_X(x) = g(x - m, \sigma), \quad m \in \mathbb{R}, \quad \sigma > 0 .$$

$$E\{X(\omega)\} = \int_{-\infty}^{\infty} x\, p_X(x)\, dx = m$$

$$\sigma_X^2 = E\{(X(\omega) - E\{X(\omega)\})^2\} = \sigma^2$$

If N independent random variables X_i are considered and

$$Y_N(\omega) = \frac{1}{N} \sum_{i=1}^{N} X_i(\omega)$$

whatever is the distribution of X_i with finite variance then

$$F_{Y_N}(y) \rightarrow F_Y(y), \; Y \text{ gaussian,}$$

$$\text{as } N \rightarrow \infty$$

(this is known as the CENTRAL LIMIT theorem). Therefore, the gaussian distribution is important in modeling a process which results from several indipendent causes, each contributing an error or influence in a different way.