Bachelor’s degree in Bioinformatics

Limits of Functions

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Sequences

- Some functions with domain **N** (the set of natural numbers) are

$$f : \mathbf{N} \to \mathbf{R}$$
$$n \to f(n)$$

- They are called **sequences,** the generic element is often written a_n and the whole sequence is often written $\{a_n\}$

- **Example:** $f(n) = \dfrac{1}{n}$
  - its values are: 1, 1/2, 1/3, 1/4, 1/5, etc.

  - When we increase n to infinity, we want "the limit of the sequence a_n as n goes to infinity" and use the notation

$$\lim_{n \to \infty} a_n$$

- **Example:** $\lim_{n \to \infty} \dfrac{1}{n}$ is 0 (it gets smaller as n increases)

Limits of a Sequence

- **Formal Definition of Limits** The sequence $\{a_n\}$ has **limit** **L**, written as

$$\lim_{n \to \infty} a_n = \mathbf{L} \qquad \text{or} \qquad \{a_n\} \to \mathbf{L}$$

if, **for every** $\varepsilon > 0$, **there exists** an integer N (**depending on** ε) such that $|a_n - \mathbf{L}| < \varepsilon$ whenever $n > N$

- If the limit exists, the sequence is called **convergent** and we say that a_n **converges** to **L** as n tends to infinity. If the sequence has no limit, it is called **divergent**

In other words, if we have a value over which the sequence "stabilizes", we say that it converges to that value, otherwise it does not converge

This structure of formal definition, with minor changes, can be used for all other limits

Limits of a Sequence

- A sequence $\{a_n\}$ converges to $+\infty$, or diverges to $+\infty$, or goes to $+\infty$, written

$$\lim_{n \to \infty} a_n = +\infty \quad \text{or} \quad \{a_n\} \to +\infty$$

if, for every M > 0, there exists an integer N (depending on M) such that $a_n > M$ whenever $n > N$

- A sequence $\{a_n\}$ converges to $-\infty$, or diverges to $-\infty$, or goes to $-\infty$, written

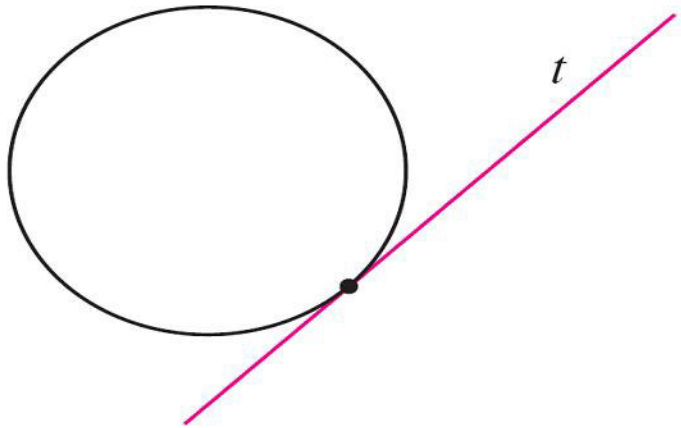
$$\lim_{n \to \infty} a_n = -\infty \quad \text{or} \quad \{a_n\} \to -\infty$$

if, for every M > 0, there exists an integer N (depending on M) such that $a_n < -M$ whenever $n > N$

We will now see limits for generic functions

The Tangent Problems

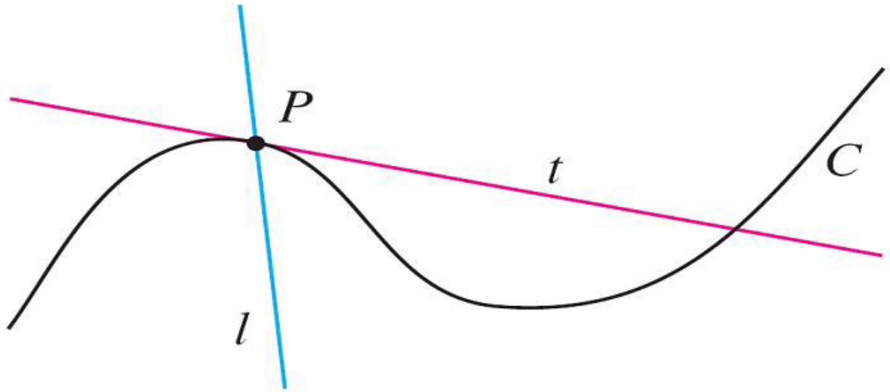
- A tangent to a curve is a line that touches the curve
- For a circle, we could simply follow Euclid and say that a tangent is a line that intersects the circle once and only once
- However, for more complicated curves, this definition may be inadequate



The Tangent Problems

- Consider two lines *l* and *t* passing through a point *P* on a curve *C*

- The line *l* intersects *C* only once, but it certainly does not look like what we think of as a tangent



- The line *t*, on the other hand, looks like a tangent but it intersects *C* twice

Example

- Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$

Solution:

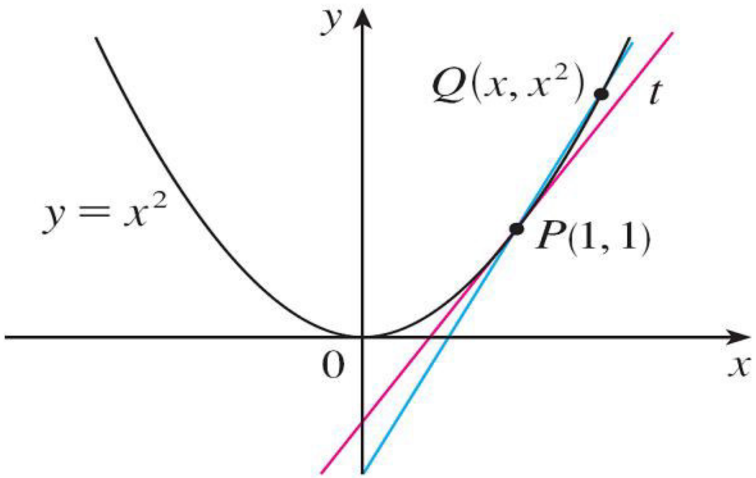
- We can find an equation of the tangent line t as soon as we know its slope m

- The difficulty is that we know only one point, P, on t, whereas we need **two points** to compute the slope

Example – *Solution*

cont'd

- We can compute an approximation to m by choosing a nearby point $Q(x, x^2)$ on the parabola and computing the slope m_{PQ} of the secant line PQ
- A **secant line** is a line that intersects a curve more than once



Example – Solution

cont'd

- We choose $x \neq 1$ so that $Q \neq P$. Then

$$m_{PQ} = \frac{y_Q - y_P}{x_Q - x_P} \Longrightarrow m_{PQ} = \frac{x^2 - 1}{x - 1}$$

- For instance, for the point $Q(1.5, 2.25)$ we have

$$m_{PQ} = \frac{2.25 - 1}{1.5 - 1}$$

$$= \frac{1.25}{0.5}$$

$$= 2.5$$

Example – Solution

cont'd

The following tables show the values of m_{PQ} for several values of x close to 1

The closer Q is to P, the closer x is to 1 and, it appears from the tables, the closer m_{PQ} is to 2

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

Example – *Solution*

cont'd

- This suggests that the slope of the tangent line t should be $m = 2$

- We say that the slope of the tangent line is the *limit* of the slopes of the secant lines, and we express this symbolically by writing

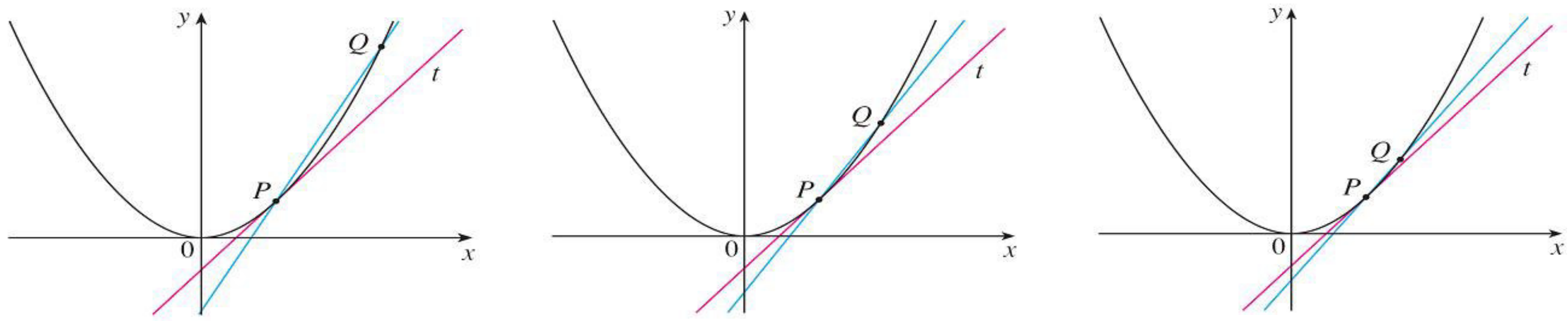
$$\lim_{Q \to P} m_{PQ} = m \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

- Assuming that the slope of the tangent line is indeed 2, we use the point-slope form of the equation of a line to write the equation of the tangent line through (1, 1) as

$$y - 1 = 2(x - 1) \qquad \text{or} \qquad y = 2x - 1$$

Example – Solution

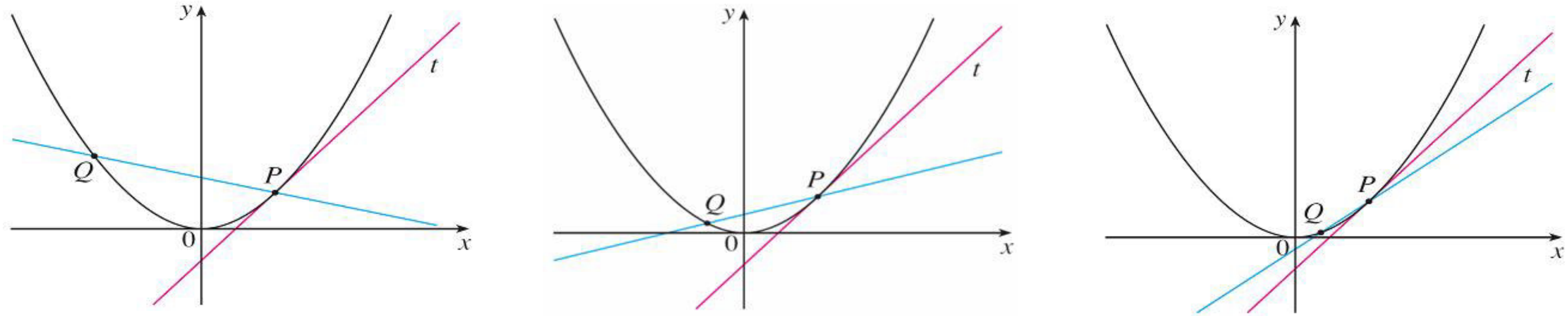
cont'd



Q approaches *P* from the right

Example – Solution

cont'd



Q approaches *P* from the left

- As *Q* approaches *P* along the parabola, the corresponding secant lines rotate about *P* and approach the tangent line *t*

Example: velocity

- Suppose that a ball is dropped from a very high building. Find the velocity of the ball after 5 seconds

- Solution:

Through experiments carried out four centuries ago, Galileo discovered that the distance fallen by any freely falling body is proportional to the square of the time it has been falling (by neglecting air resistance!)

Example velocity – *Solution* cont'd

- If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then Galileo's law is expressed by the equation

$$s(t) = 4.9t^2$$

- But this rule give us the distance, we want the velocity which is distance/time
- The velocity is clearly not constant, since the ball is accelerating. We need the instantaneous velocity

Example velocity – *Solution*

cont'd

- However, we can approximate the desired quantity by computing the average velocity over the brief time interval of a tenth of a second from t = 5 to t = 5.1

$$\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}}$$

$$= \frac{s(5.1) - s(5)}{0.1}$$

$$= \frac{4.9(5.1)^2 - 4.9(5)^2}{0.1}$$

$$= 49.49 \text{ m/s}$$

Example velocity – *Solution* cont'd

- The following table shows the results of similar calculations of the average velocity over successively smaller time periods

Time interval	Average velocity (m/s)
$5 \leqslant t \leqslant 6$	53.9
$5 \leqslant t \leqslant 5.1$	49.49
$5 \leqslant t \leqslant 5.05$	49.245
$5 \leqslant t \leqslant 5.01$	49.049
$5 \leqslant t \leqslant 5.001$	49.0049

Example velocity – *Solution*

cont'd

- It appears that as we shorten the time period, the average velocity is becoming closer to 49 m/s

- The **instantaneous velocity** when $t = 5$ is defined to be the limiting value of these average velocities over shorter and shorter time periods that start at $t = 5$

- Thus the (instantaneous) velocity after 5 s is

$$v = 49 \text{ m/s}$$

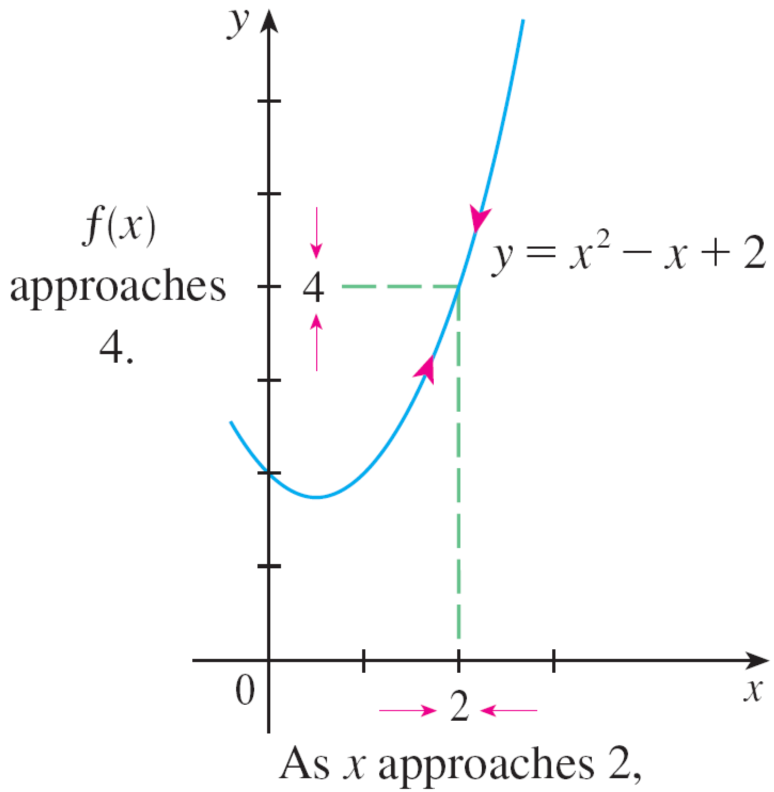
What is the limit of a function?

- Let's investigate the behavior of an example function *f* defined by $f(x) = x^2 - x + 2$ for values of *x* near 2
- for values of *x* close to 2 (but not equal to 2) we have

x	$f(x)$	x	$f(x)$
1.0	2.000000	3.0	8.000000
1.5	2.750000	2.5	5.750000
1.8	3.440000	2.2	4.640000
1.9	3.710000	2.1	4.310000
1.95	3.852500	2.05	4.152500
1.99	3.970100	2.01	4.030100
1.995	3.985025	2.005	4.015025
1.999	3.997001	2.001	4.003001

What is the limit of a function?

- From the table and the graph of *f* (a parabola) we see that, when *x* is close to 2 (on either side of 2), *f* (*x*) is close to 4



What is the limit of a function?

- In fact, it appears that we can make the values of $f(x)$ as close as we like to 4 by taking x sufficiently close to 2.

- We express this by saying "the limit of the function
- $f(x) = x^2 - x + 2$ as x approaches 2 is equal to 4"

- The notation for this is

$$\lim_{x \to 2} (x^2 - x + 2) = 4$$

Limit of a function

1 Definition We write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

2 Definition Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

Limit of a function

- An alternative notation for $\lim_{x \to a} f(x) = L$

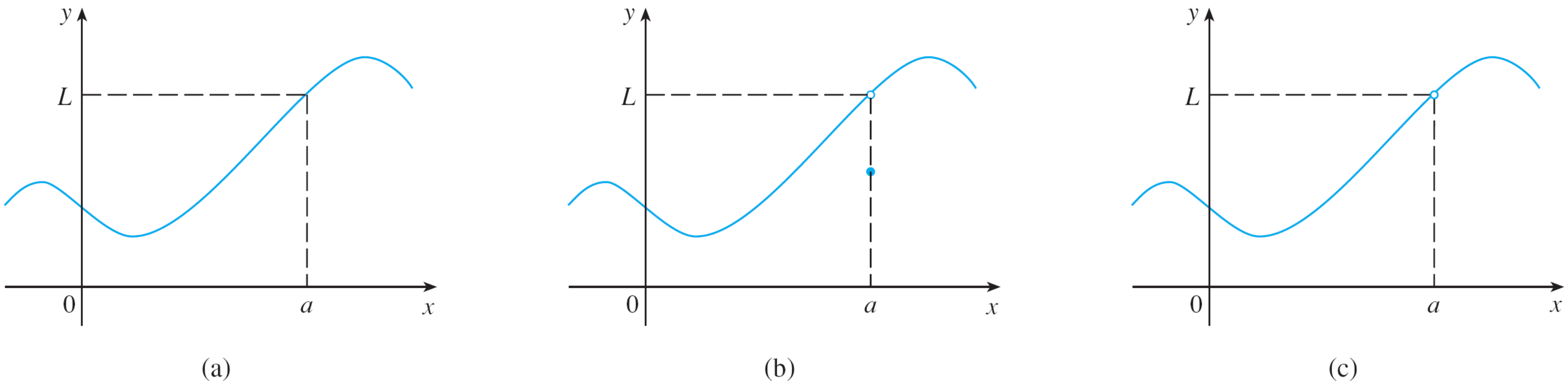
is $f(x) \to L$ as $x \to a$

which is usually read "*f* (*x*) approaches *L* as *x* approaches *a*"

- Notice "but $x \neq a$" in the definition of limit. This means that in finding the limit of *f* (*x*) as *x* approaches *a*, **we never consider *x* = *a***. In fact, *f* (*x*) need not even be defined when *x* = *a*. The only thing that matters is how *f* is defined *near* *a*

Limit of a function

- The graphs of three functions. Note that in part (c), $f(a)$ is not defined and in part (b), $f(a) \neq L$
- But in each case, regardless of what happens at a, it is true that $\lim_{x \to a} f(x) = L$



$\lim_{x \to a} f(x) = L$ in all three cases

EXAMPLE

- Guess the value of $\lim_{x \to 1} \dfrac{x - 1}{x^2 - 1}$.

- Solution:

  Notice that the function $f(x) = (x - 1)/(x^2 - 1)$ is not defined when $x = 1$, but that doesn't matter because the definition of $\lim_{x \to a} f(x)$ says that we consider values of x that are close to a but NOT equal to a

Example – Solution

cont'd

- The tables below give values of $f(x)$ (correct to six decimal places) for values of x that approach 1 (but are not equal to 1)

$x < 1$	$f(x)$
0.5	0.666667
0.9	0.526316
0.99	0.502513
0.999	0.500250
0.9999	0.500025

$x > 1$	$f(x)$
1.5	0.400000
1.1	0.476190
1.01	0.497512
1.001	0.499750
1.0001	0.499975

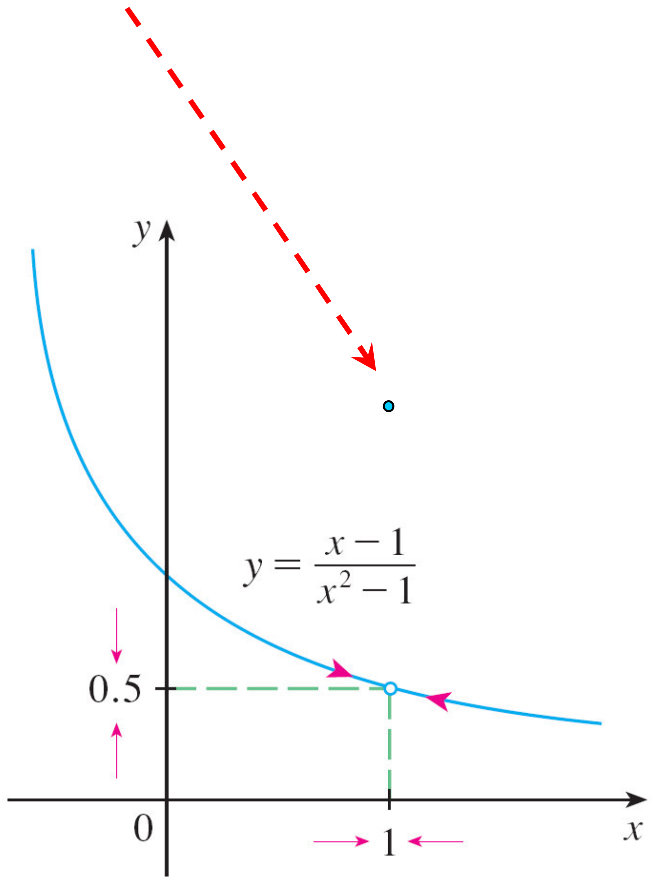
- On the basis of the values in the tables, we make the guess that

$$\lim_{x \to 1} \frac{x - 1}{x^2 - 1} = 0.5$$

Example – *Solution*

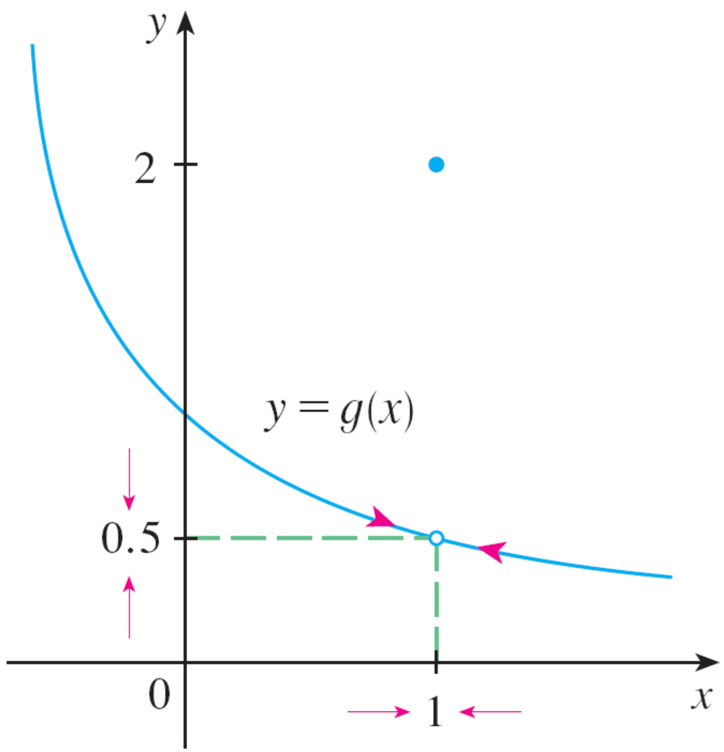
- Now let's change *f* slightly by giving it the value 2 when $x = 1$ and calling the resulting function *g*:

$$g(x) = \begin{cases} \dfrac{x-1}{x^2-1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$$



Example – *Solution*

- This new function g still has the same limit as x approaches 1



One-sided limits

- Consider a function *H* defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geqslant 0 \end{cases}$$

- *H* (*t*) approaches 0 as *t* approaches 0 from the left and *H* (*t*) approaches 1 as *t* approaches 0 from the right.

- We indicate this situation symbolically by writing

$$\lim_{t \to 0^-} H(t) = 0 \qquad \lim_{t \to 0^+} H(t) = 1$$

- "$t \to 0^-$" means that we consider only values of *t* that are less than 0
- "$t \to 0^+$" means that we consider only values of *t* that are greater than 0

Limit from right

- Given a function $f : (x',b) \to \mathbf{R}$ we say that its limit when x tends to x' from right is **L** (or that f tends to **L** when x tends to x' from the right), written

$$\lim_{x \to x'+} f(x) = \mathbf{L} \quad \text{or} \quad f(x) \underset{x \to x'+}{\to} \mathbf{L}$$

if, for every $\varepsilon > 0$, there exists a $\delta > 0$ (depending on ε) such that $|f(x) - \mathbf{L}| < \varepsilon$ whenever $x \in [x', x'+ \delta)$

- In other words, we approach x' only from the right, what happens on x' or on its left doesn't matter

Limit from left

- Given a function $f : (a, x') \to \mathbf{R}$ we say that its limit when x tends to x' from left is **L** (or that f tends to **L** when x tends to x' from the left), written

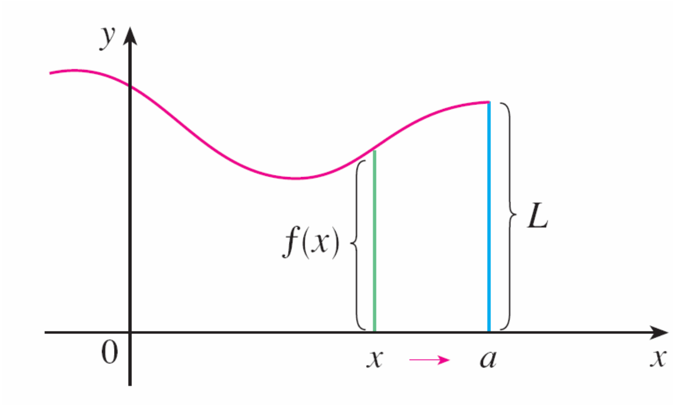
$$\lim_{x \to x'-} f(x) = \mathbf{L} \qquad \text{or} \qquad f(x) \underset{x \to x'-}{\to} \mathbf{L}$$

if, for every $\varepsilon > 0$, there exists a $\delta > 0$ (depending on ε) such that $|f(x) - \mathbf{L}| < \varepsilon$ whenever $x \in (x'- \delta, x']$

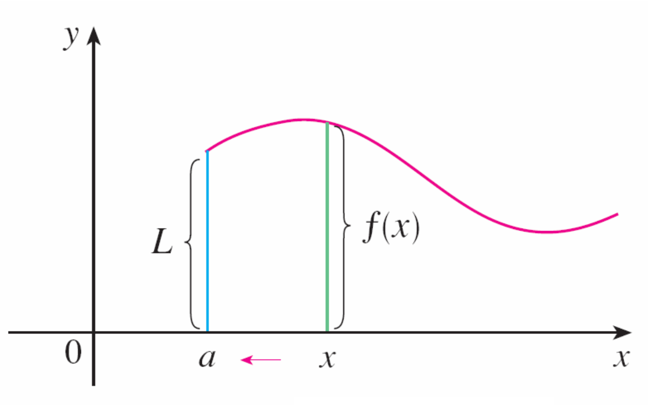
- Here we approach x' only from the left, what happens on x' or on its right doesn't matter

One-sided limits

- Approching from left or from right is illustrated below



(a) $\lim_{x \to a^-} f(x) = L$



(b) $\lim_{x \to a^+} f(x) = L$

Limit vs. One-sided limits

- By comparing the definition of limit of a function with the definitions of one-sided limits, we see that

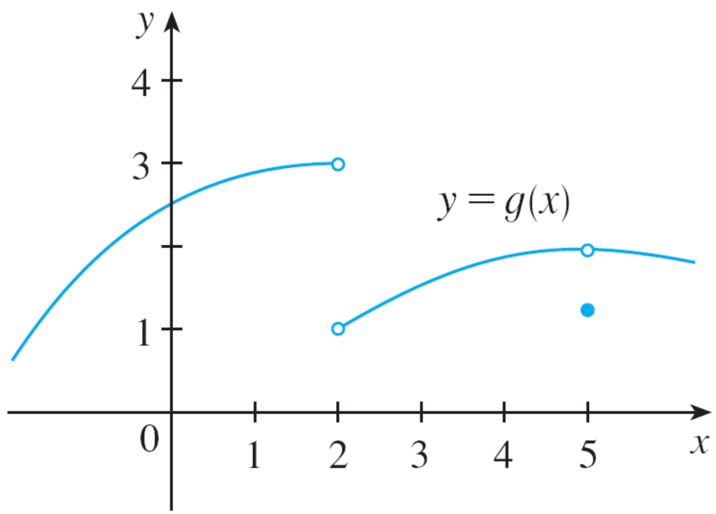
$$\boxed{3} \qquad \lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L$$

Example

- The graph of a function *g* is given below. Use it to find the values (if they exist) of the following:

(a) $\lim_{x \to 2^-} g(x)$ (b) $\lim_{x \to 2^+} g(x)$ (c) $\lim_{x \to 2} g(x)$

(d) $\lim_{x \to 5^-} g(x)$ (e) $\lim_{x \to 5^+} g(x)$ (f) $\lim_{x \to 5} g(x)$



Example – *Solution*

- From the graph we see that the values of $g(x)$ approach 3 as x approaches 2 from the left, but they approach 1 as x approaches 2 from the right.

- Therefore

$$\text{(a)} \quad \lim_{x \to 2^-} g(x) = 3 \qquad\qquad \text{(b)} \quad \lim_{x \to 2^+} g(x) = 1$$

- (c) Since the left and right limits are different, we conclude that $\lim_{x\to 2} g(x)$ **does not exist**

Example – *Solution*

cont'd

- The graph also shows that

(d) $\lim_{x \to 5^-} g(x) = 2$

(e) $\lim_{x \to 5^+} g(x) = 2$

- (f) This time the left and right limits are the same and so, we have

$$\lim_{x \to 5} g(x) = 2$$

- Despite this fact, notice that $g(5) \neq 2$

Infinite limits

4 Definition Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

- Another notation for $\lim_{x \to a} f(x) = \infty$ is

$$f(x) \to \infty \quad \text{as} \quad x \to a$$

Infinite limits

- Again, the symbol ∞ is not a number, but the expression $\lim_{x \to a} f(x) = \infty$ is often read as

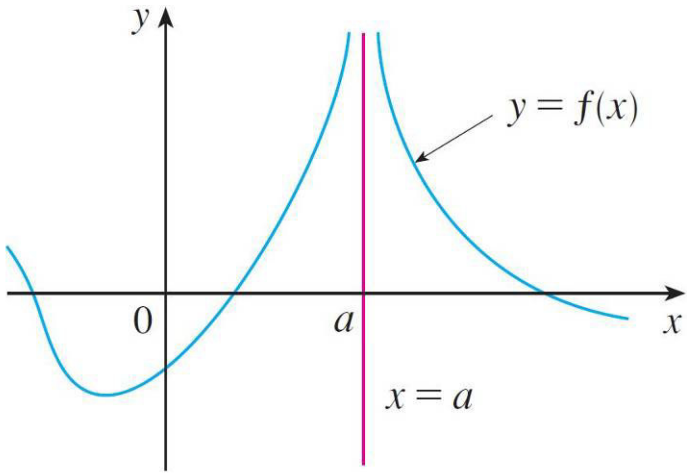
- "the limit of $f(x)$, as x approaches a, is infinity"

- or "$f(x)$ becomes infinite as x approaches a"

- or "$f(x)$ increases without bound as x approaches a"

Infinite limits

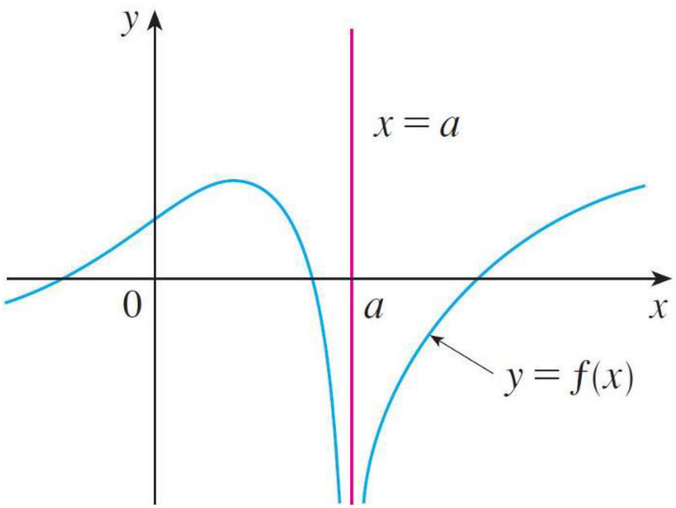
- This definition can be illustrated graphically as



$$\lim_{x \to a} f(x) = \infty$$

Infinite limits

- A similar sort of limit, for functions that become large negative as *x* gets close to *a*, is



$$\lim_{x \to a} f(x) = -\infty$$

Infinite limits

5 Definition Let f be defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

- The symbol $\lim_{x \to a} f(x) = -\infty$ can be read as "the limit of $f(x)$, as x approaches a, is negative infinity" or "$f(x)$ decreases without bound as x approaches a." As an example we have

$$\lim_{x \to 0} \left(-\frac{1}{x^2} \right) = -\infty$$

Infinite limits

- Similar definitions can be given for the one-sided infinite limits

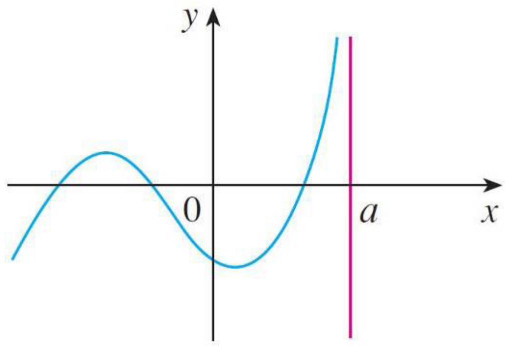
$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

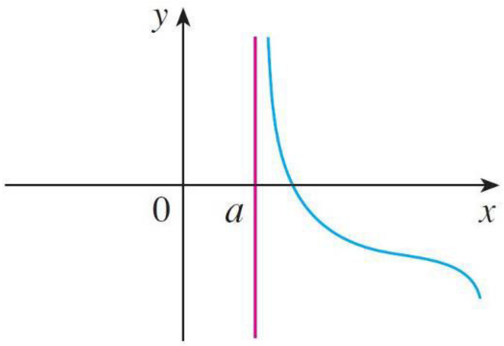
- remembering that "$x \to a^-$" means that we consider only values of x that are less than a, and similarly "$x \to a^+$" means that we consider only $x > a$

Infinite limits

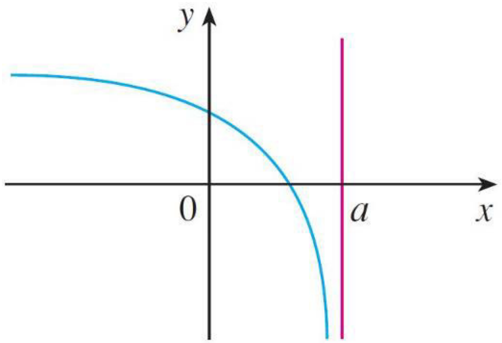
- Illustrations of those four cases are given here



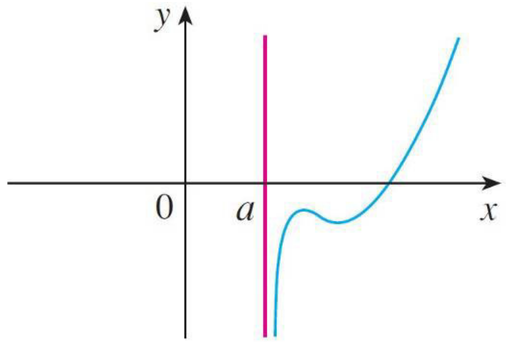
(a) $\lim_{x \to a^-} f(x) = \infty$



(b) $\lim_{x \to a^+} f(x) = \infty$



(c) $\lim_{x \to a^-} f(x) = -\infty$



(d) $\lim_{x \to a^+} f(x) = -\infty$

Vertical Asymptotes

6 Definition The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

Limits at infinity

- Given a function $f : [a,+\infty) \to \mathbf{R}$ we say that its limit when x tends to +infinity is **L** (or that f tends to **L** when x tends to +infinity), written

$$\lim_{x \to +\infty} f(x) = \mathbf{L} \qquad \text{or} \qquad f(x) \underset{x \to +\infty}{\longrightarrow} \mathbf{L}$$

if, for every $\varepsilon > 0$, there exists a $\Delta > 0$ (depending on ε) such that $|f(x) - \mathbf{L}| < \varepsilon$ whenever $x > \Delta$

Limits at infinity

- Given a function $f : [a,+\infty) \longrightarrow \mathbf{R}$ we say that its limit when x tends to $+\infty$ is $+\infty$ (or that f tends to $+\infty$ when x tends to $+\infty$), written

$$\lim_{x \to +\infty} f(x) = +\infty \quad \text{or} \quad f(x) \underset{x \to +\infty}{\longrightarrow} +\infty$$

if, for every M > 0, there exists a $\Delta > 0$ (depending on M) such that
f(x) > M whenever $x > \Delta$

Limits at infinity

- Given a function $f : [a,+\infty) \to \mathbf{R}$ we say that its limit when x tends to $+\infty$ is $-\infty$ (or that f tends to $-\infty$ when x tends to $+\infty$), written

$$\lim_{x \to +\infty} f(x) = -\infty \quad \text{or} \quad f(x) \underset{x \to +\infty}{\longrightarrow} -\infty$$

if, for every M > 0, there exists a $\Delta > 0$ (depending on M) such that

$f(x) < -M$ whenever $x > \Delta$

Limits at infinity

- Given a function $f : (-\infty, b] \to \mathbf{R}$ we say that its limit when x tends to $-\infty$ is **L** (or that f tends to **L** when x tends to $-\infty$), written

$$\lim_{x \to -\infty} f(x) = \mathbf{L} \qquad \text{or} \qquad f(x) \underset{x \to -\infty}{\longrightarrow} \mathbf{L}$$

if, for every $\varepsilon > 0$, there exists a $\Delta > 0$ (depending on ε) such that

$|f(x) - \mathbf{L}| < \varepsilon$ whenever $x < -\Delta$

Limits at infinity

- Given a function $f : (-\infty, b] \rightarrow \mathbf{R}$ we say that its limit when x tends to $-\infty$ is $+\infty$ (or that f tends to $+\infty$ when x tends to $-\infty$), written

$$\lim_{x \to -\infty} f(x) = +\infty \quad \text{or} \quad f(x) \underset{x \to -\infty}{\rightarrow} +\infty$$

if, for every M > 0, there exists a $\Delta > 0$ (depending on M) such that

$f(x) > M$ whenever $x < -\Delta$

Limits at infinity

- Given a function $f : (-\infty, b] \to \mathbf{R}$ we say that its limit when x tends to $-\infty$ is $-\infty$ (or that f tends to $-\infty$ when x tends to $-\infty$), written

$$\lim_{x \to -\infty} f(x) = -\infty \quad \text{or} \quad f(x) \underset{x \to -\infty}{\longrightarrow} -\infty$$

if, for every M > 0, there exists a $\Delta > 0$ (depending on M) such that

$f(x) < -M$ whenever $x < -\Delta$

Example

- Find the vertical asymptotes of $f(x) = \tan x$

- Solution:
  Since $$\tan x = \frac{\sin x}{\cos x}$$

- there are potential vertical asymptotes where $\cos x = 0$

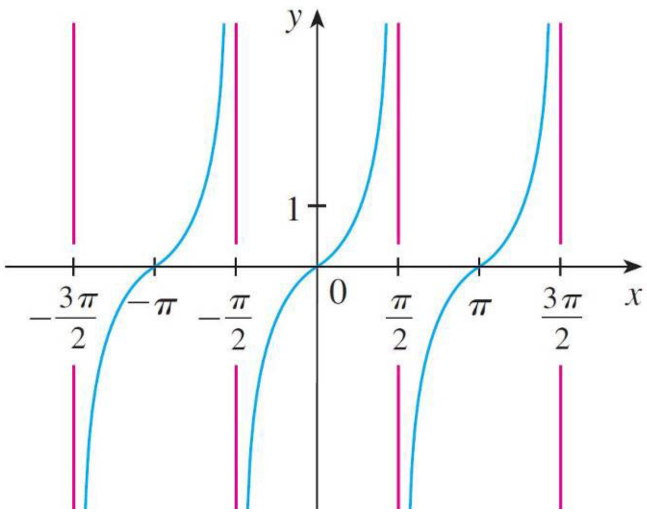
- In fact, since $\cos x \to 0^+$ as $x \to (\pi/2)^-$ and $\cos x \to 0^-$ as $x \to (\pi/2)^+$, whereas $\sin x$ is positive when x is near $\pi/2$, we have

$$\lim_{x\to(\pi/2)^-} \tan x = \infty \qquad \text{and} \qquad \lim_{x\to(\pi/2)^+} \tan x = -\infty$$

Example – Solution

cont'd

- This shows that the line $x = \pi/2$ is a vertical asymptote. Similar reasoning shows that the lines $x = (2n + 1)\pi/2$, where n is an integer, are all vertical asymptotes of $f(x) = \tan x$



$y = \tan x$

Limits Laws to compute limits

- we use the following properties of limits, called the *Limit Laws,* to calculate limits

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \qquad \text{and} \qquad \lim_{x\to a} g(x)$$

exist. Then

1. $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

2. $\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

3. $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x)$

4. $\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{if } \lim_{x\to a} g(x) \neq 0$

Limits Laws to compute limits

These five laws can be stated verbally as follows:

- **Sum Law**

1. The limit of a sum is the sum of the limits

- **Difference Law**

2. The limit of a difference is the difference of the limits

- **Constant Multiple Law**

3. The limit of a constant times a function is the constant times the limit of the function

- **Product Law**

4. The limit of a product is the product of the limits

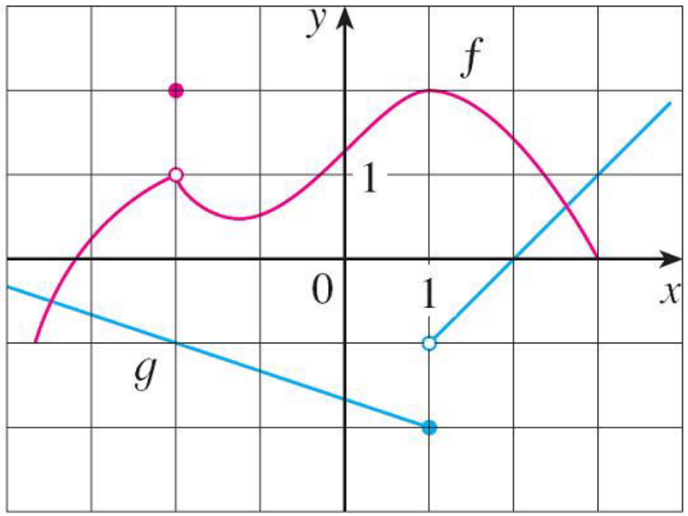
- **Quotient Law**

5. The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0)

Example

- Use the Limit Laws and the graphs of f and g given here to evaluate the following limits, if they exist

(a) $\lim_{x \to -2} [f(x) + 5g(x)]$ (b) $\lim_{x \to 1} [f(x)g(x)]$ (c) $\lim_{x \to 2} \frac{f(x)}{g(x)}$



Example – Solution (A)

- From the graphs of f and g we see that
- $\lim_{x \to -2} f(x) = 1$ and $\lim_{x \to -2} g(x) = -1$
- Therefore we have

$$\begin{aligned}
\lim_{x \to -2} [f(x) + 5g(x)] &= \lim_{x \to -2} f(x) + \lim_{x \to -2} [5g(x)] && \text{(by Law 1)} \\
&= \lim_{x \to -2} f(x) + 5 \lim_{x \to -2} g(x) && \text{(by Law 3)} \\
&= 1 + 5(-1) \\
&= -4
\end{aligned}$$

Example – *Solution (b)* cont'd

- We see that $\lim_{x \to 1} f(x) = 2$. But $\lim_{x \to 1} g(x)$ does not exist because the left and right limits are different:

$$\lim_{x \to 1^-} g(x) = -2 \qquad \lim_{x \to 1^+} g(x) = -1$$

- So we can't use Law 4 for the desired limit. But we *can* use Law 4 for the one-sided limits:

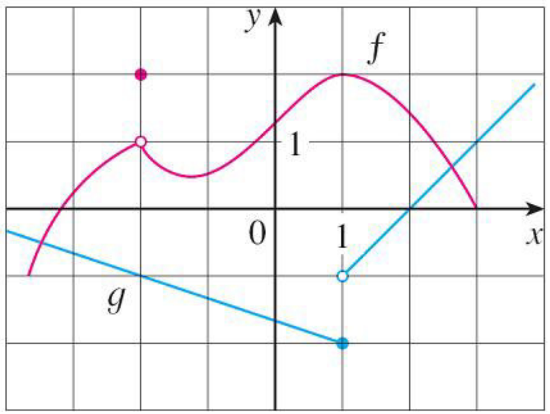
$$\lim_{x \to 1^-} [f(x)g(x)] = 2 \cdot (-2) = -4 \qquad \lim_{x \to 1^+} [f(x)g(x)] = 2 \cdot (-1) = -2$$

- The left and right limits aren't equal, so $\lim_{x \to 1} [f(x)g(x)]$ does not exist.

Example – Solution (c)

cont'd

- The graphs show that
- $\lim_{x \to 2} f(x) \approx 1.4$ and $\lim_{x \to 2} g(x) = 0$



- Since the limit of the denominator is 0, we can't use Law 5.
- The given limit does not exist because the denominator approaches 0 and the sign is different if from right or from left, while the numerator approaches a nonzero number. We can only consider the limits from right or from left.

Limits Laws to compute limits

- If we use the Product Law repeatedly with $g(x) = f(x)$, we obtain the following law

Power Law

6. $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ where n is a positive integer

- In applying these six limit laws, we also need to use two basic limits:

7. $\lim_{x \to a} c = c$	**8.** $\lim_{x \to a} x = a$

- These limits are obvious from an intuitive point of view (state them in words or draw graphs of $y = c$ and $y = x$)

Limits Laws to compute limits

- If we now put $f(x) = x$ in Law 6 and use Law 8, we get another useful special limit

9. $\lim_{x \to a} x^n = a^n$ where n is a positive integer

- A similar limit holds for roots as follows

10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

(If n is even, we assume that $a > 0$.)

- More generally, we have the following law

Root Law

11. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is a positive integer

$\left[\text{If } n \text{ is even, we assume that } \lim_{x \to a} f(x) > 0.\right]$

Limits Laws to compute limits

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

- Functions with the Direct Substitution Property are called *continuous at a*

- We can also use the following property

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$, provided the limits exist.

- Another way to compute limits is by computing both left and right limits and see if they are equal. Limits laws also hold for one-sided limits

Limits Laws to compute limits

The next two theorems give two additional properties of limits

2 Theorem If $f(x) \leqslant g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leqslant \lim_{x \to a} g(x)$$

3 The Squeeze Theorem If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

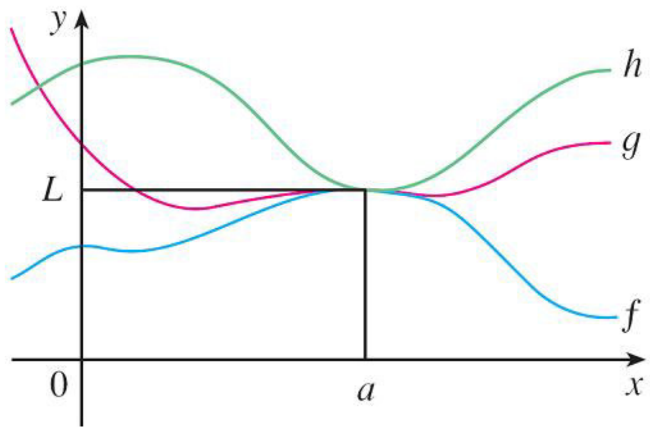
$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then
$$\lim_{x \to a} g(x) = L$$

Limits Laws to compute limits

- The Squeeze Theorem, which is sometimes called the Sandwich Theorem or the Pinching Theorem, is illustrated here

- It says that if $g(x)$ is squeezed between $f(x)$ and $h(x)$ near a, and if f and h have the same limit L at a, then g is forced to have the same limit L at a



Using the Precise Definition of Limit

To motivate the precise definition of a limit, let's consider the function

$$f(x) = \begin{cases} 2x - 1 & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$$

It is intuitively clear that, when x is close to 3 but $x \neq 3$, then $f(x)$ is close to 5, and so $\lim_{x \to 3} f(x) = 5$

To obtain more detailed information about how $f(x)$ varies when x is close to 3, we ask the following question:
How close to 3 does x have to be so that $f(x)$ differs from 5 by less than 0.1 ?

Using the Precise Definition of Limit

The distance from x to 3 is $|x - 3|$ and the distance from $f(x)$ to 5 is $|f(x) - 5|$, so our problem is to find a number δ such that

$$|f(x) - 5| < 0.1 \quad \text{if} \quad |x - 3| < \delta \qquad (\text{but } x \neq 3)$$

If $|x - 3| > 0$, then $x \neq 3$, so an equivalent formulation of our problem is to find a number δ such that

$$|f(x) - 5| < 0.1 \quad \text{if} \quad 0 < |x - 3| < \delta$$

Using the Precise Definition of Limit

If $0 < | x - 3 | < (0.1)/2 = 0.05$ then

$$| f(x) - 5 | = | (2x - 1) - 5 | = | 2x - 6 |$$

$$= 2| x - 3 | < 2(0.05) = 0.1$$

that is,

$| f(x) - 5 | < 0.1$ if $0 < | x - 3 | < 0.05$

Thus an answer to the problem is given by $\delta = 0.05$:

if x is within a distance of 0.05 from 3, then $f(x)$ will be within a distance of 0.1 from 5

Using the Precise Definition of Limit

If we ask for 0.01 instead of 0.1 in our problem, then by using the same method we find that $f(x)$ will differ from 5 by less than 0.01 if x differs from 3 by less than $(0.01)/2 = 0.005$:

$$|f(x) - 5| < 0.01 \qquad \text{if} \qquad 0 < |x - 3| < 0.005$$

Similarly,

$$|f(x) - 5| < 0.001 \qquad \text{if} \qquad 0 < |x - 3| < 0.0005$$

The numbers 0.1, 0.01 and 0.001 that we have considered may be seen as *error tolerances* that we might allow

Using the Precise Definition of Limit

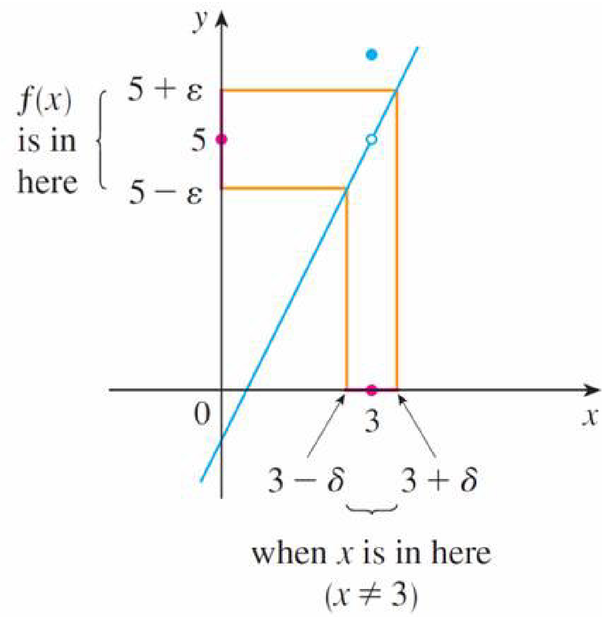
For 5 to be the precise limit of $f(x)$ as x approaches 3, we must not only be able to bring the difference between $f(x)$ and 5 below each of these three numbers; we must be able to bring it below *any* positive number

And, by the same reasoning, we can! We write ε to denote an arbitrary positive number, then we find as before that

$$|f(x) - 5| < \varepsilon \quad \text{if} \quad 0 < |x - 3| < \delta = \frac{\varepsilon}{2}$$

Using the Precise Definition of Limit

This says that $f(x)$ is close to 5 when x is close to 3 because it says that we can make the values of $f(x)$ within an arbitrary distance ε from 5 by taking the values of x within a distance $\varepsilon/2$ from 3 (but $x \neq 3$)



This can also be rewritten as:

If $3 - \delta < x < 3 + \delta \quad (x \neq 3)$

Then $5 - \varepsilon < f(x) < 5 + \varepsilon$

Using the Precise Definition of Limit

The statement "By taking the value of x ($\neq 3$) in the interval $(3 - \delta, 3 + \delta)$ we can make the value of $f(x)$ lie in the interval $(5 - \varepsilon, 5 + \varepsilon)$"

exactly corresponds to the precise definition of limit already seen

$\boxed{2}$ **Definition** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

Using the Precise Definition of Limit

In other words,

$$\lim_{x \to a} f(x) = L$$

means that the values of $f(x)$ can be made as close as we please to L by taking x close enough to a (but not equal to a)

Interval view of Limit Definition

We can also reformulate the precise definition of limit in terms of intervals by observing that the inequality $|x - a| < \delta$ is equivalent to $-\delta < x - a < \delta$, which in turn can be written as

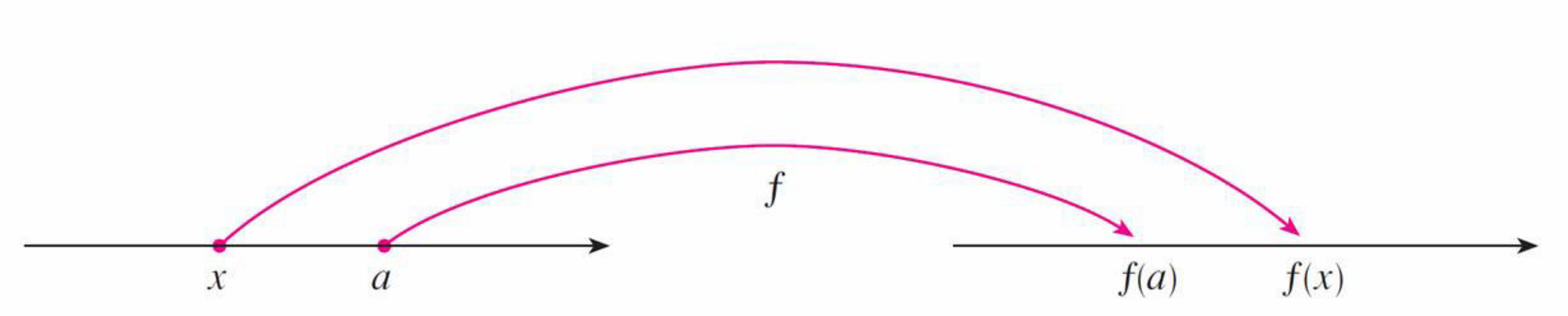
$a - \delta < x < a + \delta$

Also $0 < |x - a|$ is true if and only if $x - a \neq 0$, that is, $x \neq a$

Similarly, the inequality $|f(x) - L| < \varepsilon$ is equivalent to the pair of inequalities $L - \varepsilon < f(x) < L + \varepsilon$. Therefore, in terms of intervals, the definition means that for every $\varepsilon > 0$ (no matter how small ε is) we can find $\delta > 0$ such that if x lies in the open interval $(a - \delta, a + \delta)$ and $x \neq a$, then $f(x)$ lies in the open interval $(L - \varepsilon, L + \varepsilon)$

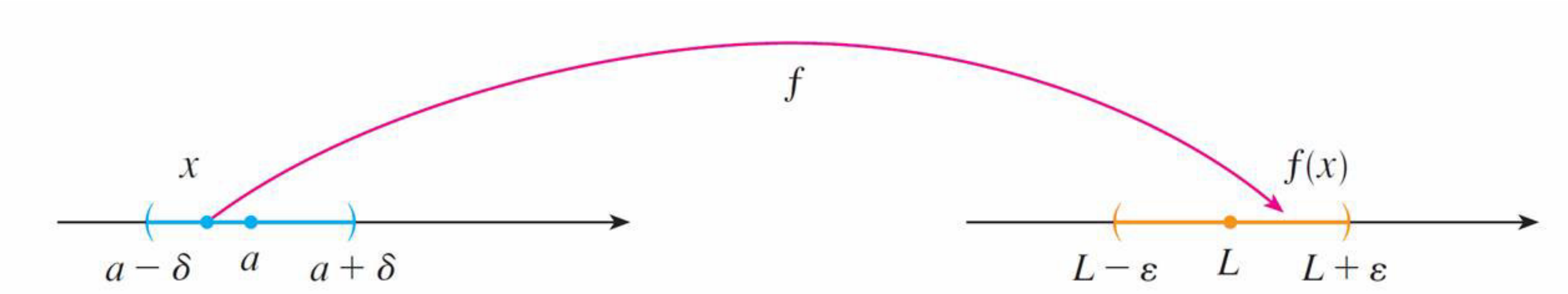
Geometric view of Limit Definition

We can view this geometrically by using the following arrow diagram of *f*



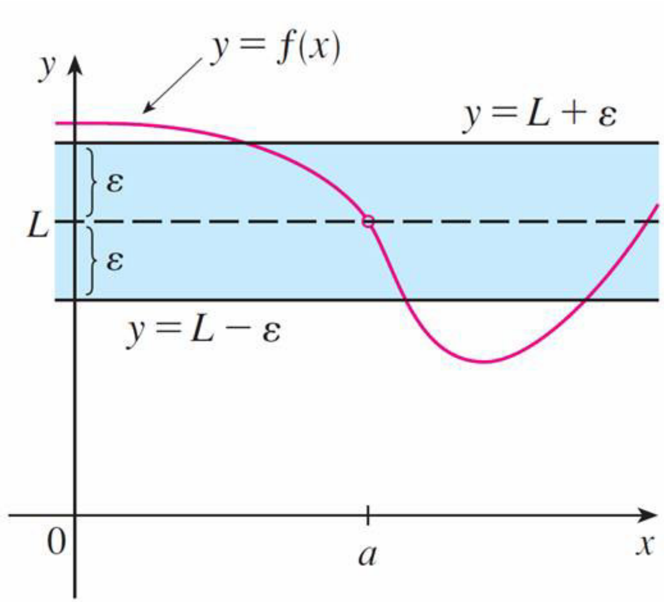
Geometric view of Limit Definition

The definition of limit says that if any small interval $(L - \varepsilon, L + \varepsilon)$ is given around L, then we can find an interval $(a - \delta, a + \delta)$ around a such that f maps all the points in $(a - \delta, a + \delta)$ (except possibly a) into the interval $(L - \varepsilon, L + \varepsilon)$



Geometric view of Limit Definition

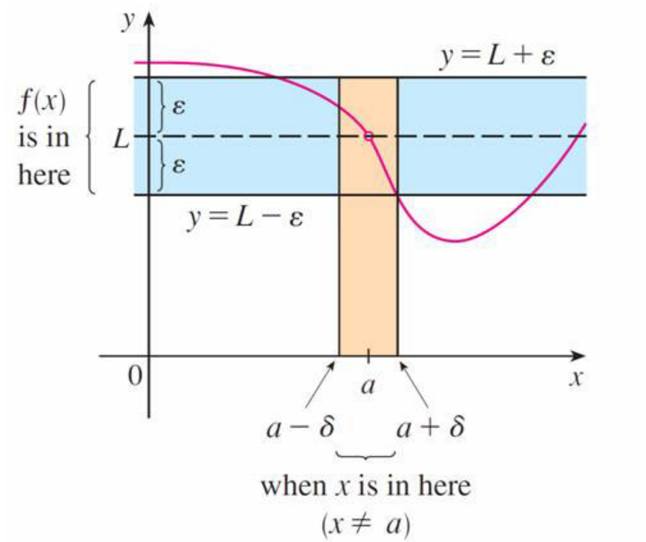
We can also view this on the graph of f. Given $\varepsilon > 0$, we draw the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ on the graph of f



Geometric view of Limit Definition

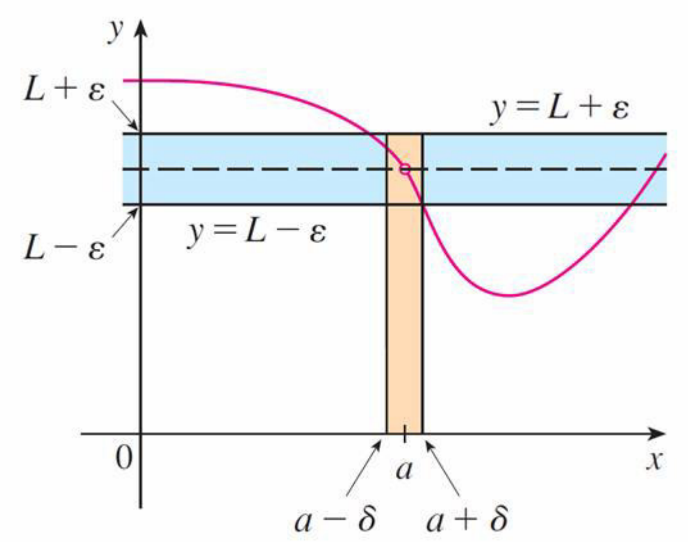
If $\lim_{x \to a} f(x) = L$, then we can find a number $\delta > 0$ such that, if we take x in the interval $(a - \delta, a + \delta)$ (but $x \neq a$), then the curve $y = f(x)$ lies between the lines $y = L - \varepsilon$ and $y = L + \varepsilon$

Note that, when δ has been found, then any smaller δ will also work



Geometric view of Limit Definition

The process must work for *every* positive number ε, no matter how small it is: if a smaller ε is chosen, we may simply need a smaller δ



Example

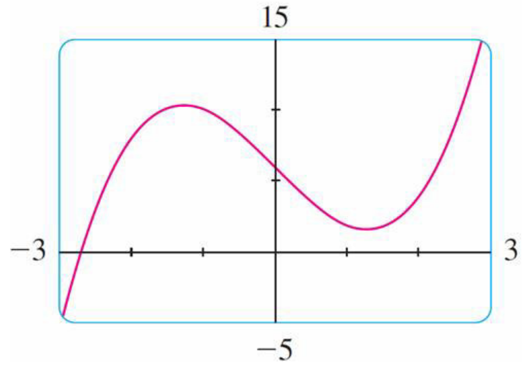
Use the graph to find a number δ such that

$$\text{if } |x - 1| < \delta \quad \text{then} \quad |(x^3 - 5x + 6) - 2| < 0.2$$

In other words, find a number δ that corresponds to $\varepsilon = 0.2$ in the definition of a limit for the function $f(x) = x^3 - 5x + 6$ with $a = 1$ and $L = 2$

Example – *Solution*

The graph of f is here; we are interested in the region near the point (1, 2)



Notice that we can rewrite the inequality

$$|(x^3 - 5x + 6) - 2| < 0.2$$

as

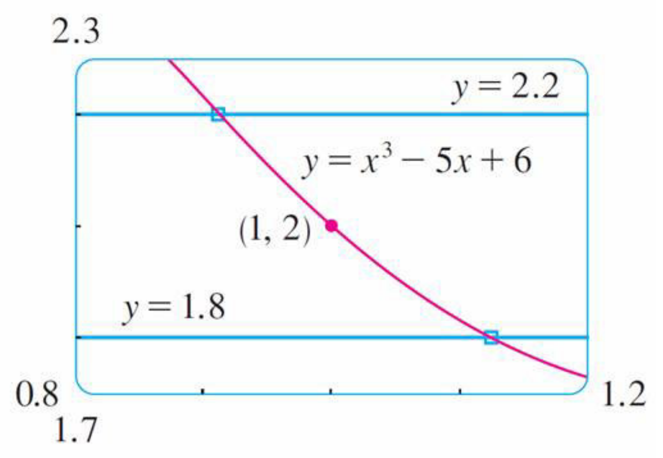
$$1.8 < x^3 - 5x + 6 < 2.2$$

Example – *Solution*

cont'd

So we need to determine the values of x for which the curve $y = x^3 - 5x + 6$ lies between the horizontal lines $y = 1.8$ and $y = 2.2$

Therefore we graph the curve $y = x^3 - 5x + 6$ and the two lines $y = 1.8$ and $y = 2.2$ near the point (1,2)



Example – *Solution*

cont'd

Then we estimate that the x-coordinate of the point of intersection of the line $y = 2.2$ and the curve $y = x^3 - 5x + 6$ is about 0.92

Similarly, $y = x^3 - 5x + 6$ intersects the line $y = 1.8$ when $x \approx 1.12$. So, we can say that

$$\text{if} \quad 0.92 < x < 1.12 \quad \text{then} \quad 1.8 < x^3 - 5x + 6 < 2.2$$

This interval $(0.92, 1.12)$ is not symmetric about $x = 1$. The distance from $x = 1$ to the left endpoint is $1 - 0.92 = 0.08$ and the distance to the right endpoint is 0.12

Example – *Solution*

cont'd

We choose δ as the smaller of these numbers: $\delta = 0.08$

Then we can rewrite our inequalities in terms of distances as follows:

$$\text{if} \quad |x - 1| < 0.08 \quad \text{then} \quad |(x^3 - 5x + 6) - 2| < 0.2$$

This just says that by keeping x within 0.08 of 1, we are able to keep $f(x)$ within 0.2 of 2

We chose $\delta = 0.08$, but any smaller positive value of δ would also have worked

Example 2

Prove that $$\lim_{x \to 3} (4x - 5) = 7.$$

Solution:

1. *Preliminary analysis of the problem (guessing a value for δ)*

   Let ε be a given positive number. We want to find a number δ such that

   if $0 < |x - 3| < \delta$ then $|(4x - 5) - 7| < \varepsilon$

   But $|(4x - 5) - 7| = |4x - 12| = |4(x - 3)| = 4|x - 3|$

Example 2 – *Solution*

cont'd

Therefore we want δ such that

if $0 < |x - 3| < \delta$ then $4|x - 3| < \varepsilon$

that is, if $0 < |x - 3| < \delta$ then $|x - 3| < \frac{\varepsilon}{4}$

This suggests that we should choose $\delta = \varepsilon/4$

Example 2 – *Solution*

cont'd

2. *Proof* (*showing that this δ works*). Given $\varepsilon > 0$, choose $\delta = \varepsilon/4$. If $0 < |x - 3| < \delta$, then

$$|(4x - 5) - 7| = |4x - 12| = 4|x - 3| < 4\delta = 4\left(\frac{\varepsilon}{4}\right) = \varepsilon$$

Thus

if $\quad 0 < |x - 3| < \delta \qquad$ then $\qquad |(4x - 5) - 7| < \varepsilon$

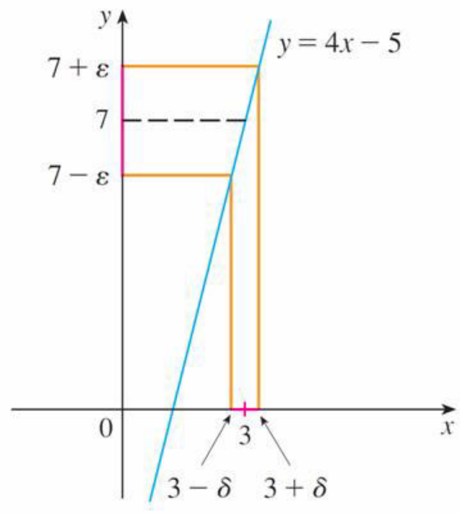
Example 2 – *Solution*

cont'd

Therefore, by the definition of a limit,

$$\lim_{x \to 3} (4x - 5) = 7$$

This example is illustrated as follows



Example 3

Prove that $$\lim_{x \to 0^+} \sqrt{x} = 0.$$

Example 3 – *Solution*

1. *Guessing a value for δ.* Let ε be a given positive number. Here $a = 0$ and $L = 0$, so we want to find a number δ such that

$$\text{if} \quad 0 < x < \delta \qquad \text{then} \qquad |\sqrt{x} - 0| < \varepsilon$$

that is,

$$\text{if} \quad 0 < x < \delta \qquad \text{then} \qquad \sqrt{x} < \varepsilon$$

or, squaring both sides of the inequality $\sqrt{x} < \varepsilon$, we get

$$\text{if} \quad 0 < x < \delta \qquad \text{then} \qquad x < \varepsilon^2$$

This suggests that we should choose $\delta = \varepsilon^2$

Example 3 – *Solution*

cont'd

2. *Showing that this δ works.* Given $\varepsilon > 0$, let $\delta = \varepsilon^2$. If $0 < x < \delta$, then

$$\sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$$

so $|\sqrt{x} - 0| < \varepsilon$

this shows that $\lim_{x \to 0^+} \sqrt{x} = 0.$

Geometric view of Infinite Limits

6 Definition Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

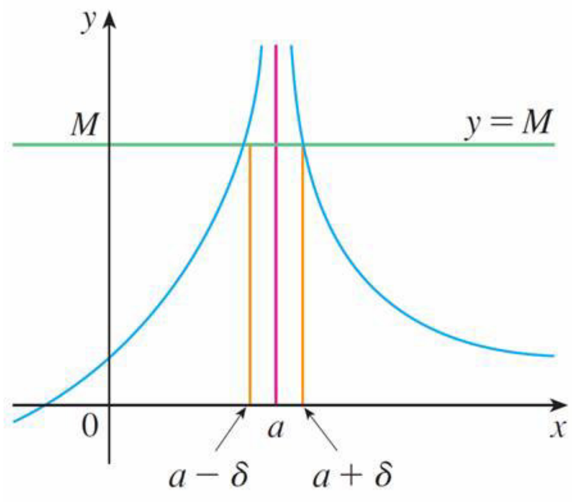
$$\lim_{x \to a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that

$$\text{if} \qquad 0 < |x - a| < \delta \qquad \text{then} \qquad f(x) > M$$

Geometric view of Infinite Limits

This says that the values of $f(x)$ can be made arbitrarily large (larger than any given number M) by taking x close enough to a (within a distance δ, where δ depends on M, but with $x \neq a$)



Given any M, we can find a $\delta > 0$ such that if x is in the interval $(a - \delta, a + \delta)$ but $x \neq a$, then the curve $y = f(x)$ lies above the line $y = M$

If a larger M is chosen, then we may need a smaller δ

Example

Prove that $$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

Solution:

Let M be a given positive number. We want to find a number δ such that

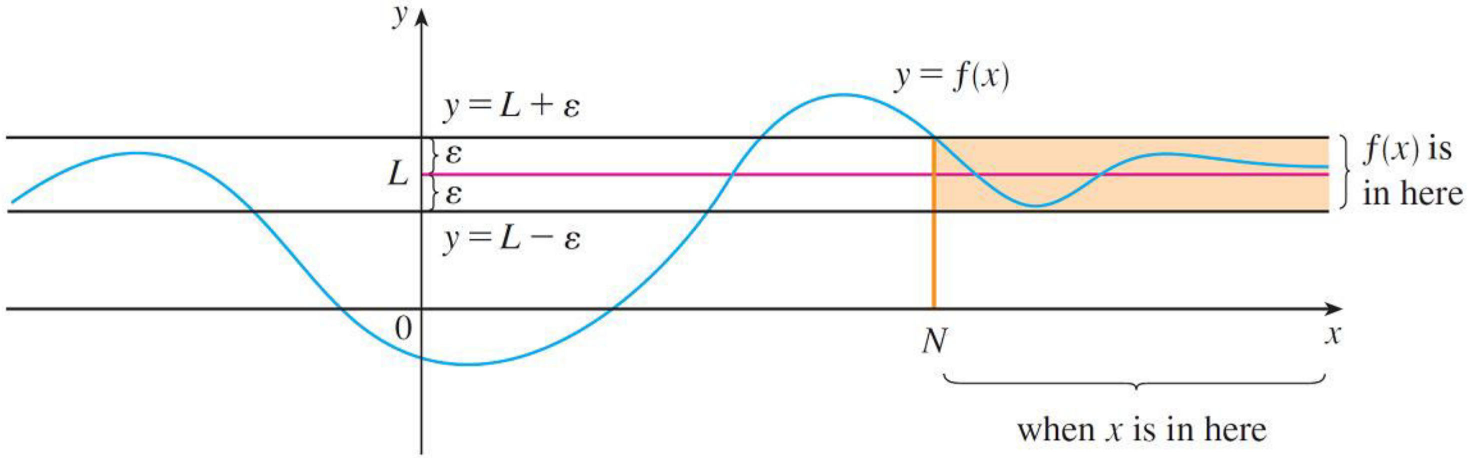
if $0 < |x| < \delta$ then $1/x^2 > M$

But $$\frac{1}{x^2} > M \iff x^2 < \frac{1}{M} \iff |x| < \frac{1}{\sqrt{M}}$$

So if we choose $\delta = 1/\sqrt{M}$ and $0 < |x| < \delta = 1/\sqrt{M}$, then $1/x^2 > M$. This shows that $1/x^2 \to \infty$ as $x \to 0$

Geometric view of Limits at infinity

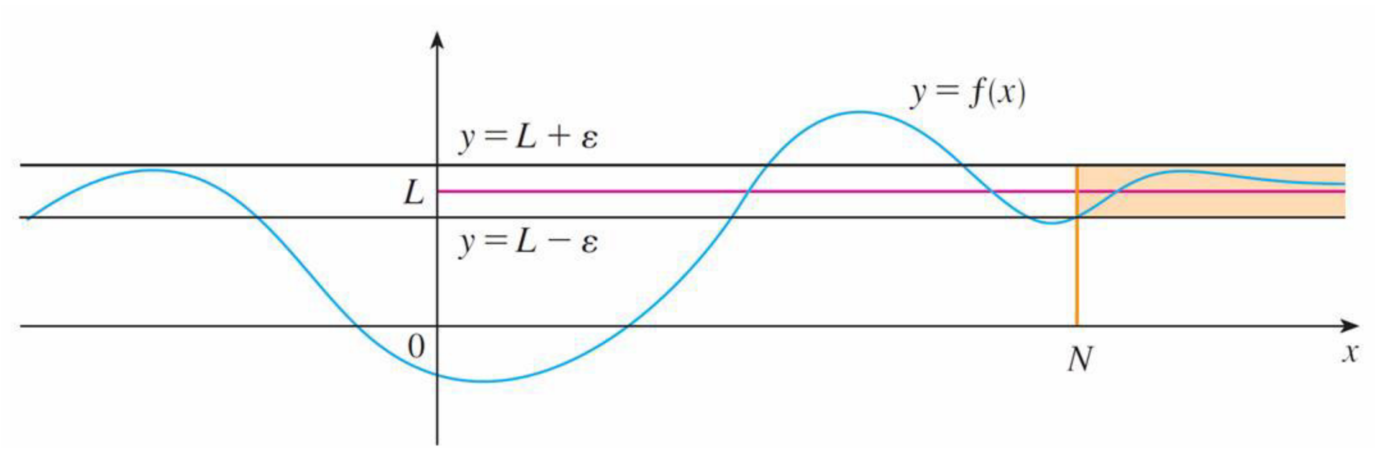
- Graphically it says that by choosing *x* large enough (larger than some number *N*) we can make the graph of *f* lie between the given horizontal lines $y = L - \varepsilon$ and $y = L + \varepsilon$



$$\lim_{x \to \infty} f(x) = L$$

Geometric view of Limits at infinity

- This must be true no matter how small we choose *ε*. If a smaller value of *ε* is chosen, then a larger value of *N* may be required



$$\lim_{x \to \infty} f(x) = L$$

Example

- Prove that $\lim_{x \to \infty} \frac{1}{x} = 0$

- Solution:

Given $\varepsilon > 0$, we want to find N such that

$$\text{if} \quad x > N \quad \text{then} \quad \left| \frac{1}{x} - 0 \right| < \varepsilon$$

In computing the limit we may assume that $x > 0$

Then $\quad 1/x < \varepsilon \iff x > 1/\varepsilon$

Example – *Solution*

cont'd

- Let's choose N = 1/ε. So

If $x > N = \dfrac{1}{\varepsilon}$ then $\left|\dfrac{1}{x} - 0\right| = \dfrac{1}{x} < \varepsilon$

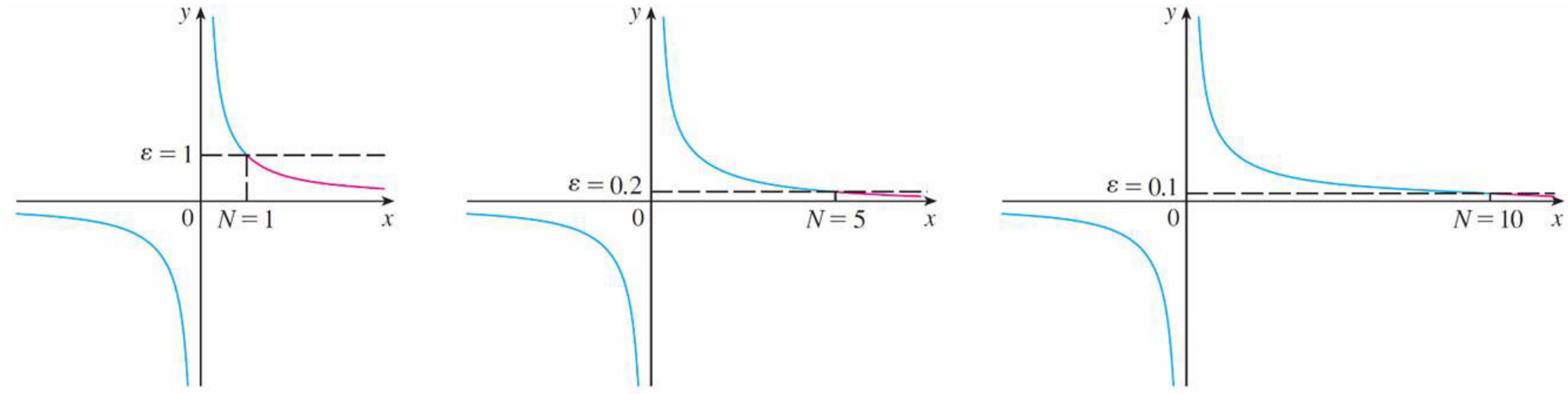
- Therefore

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Example – *Solution*

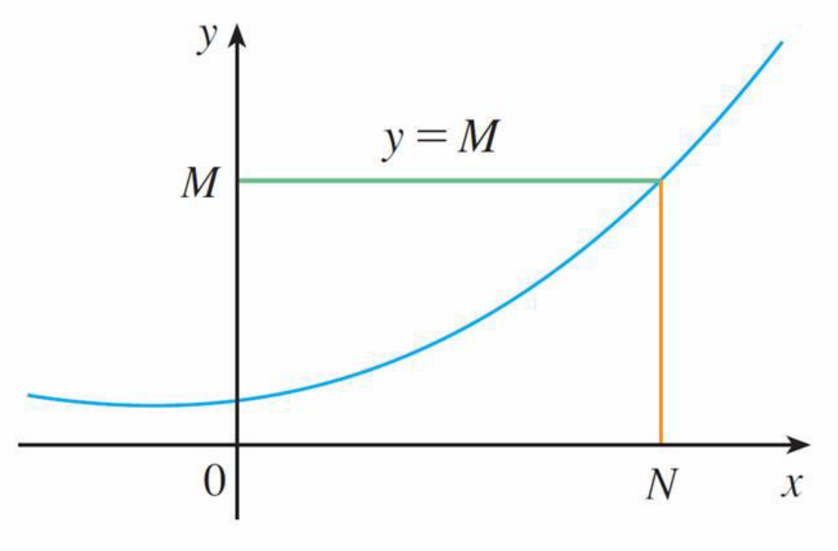
cont'd

- The figure shows some values of ε and the corresponding values of N



Geometric view of infinite limits at infinity

- Finally we note that an infinite limit at infinity can be seen as follows



$$\lim_{x \to \infty} f(x) = \infty$$

Continuity

- In some cases the limit of a function as *x* approaches *a* is simply the value of the function at *a*. Functions with this property are called *continuous at a*

- Physical phenomena are continuous more often than not
- Geometrically, you can think of a continuous function as a function whose graph has no break in it. The graph can be drawn without removing your pen from the paper

1 Definition A function f is **continuous at a number *a*** if

$$\lim_{x \to a} f(x) = f(a)$$

Continuity

- This definition of continuity requires three things:

1. $f(a)$ is defined (that is, a is in the domain of f)

2. $\lim_{x \to a} f(x)$ exists

3. $\lim_{x \to a} f(x) = f(a)$

- The definition says that f is continuous at a if $f(x)$ approaches $f(a)$ as x approaches a. Thus a continuous function f has the property that a small change in x produces only a small change in $f(x)$

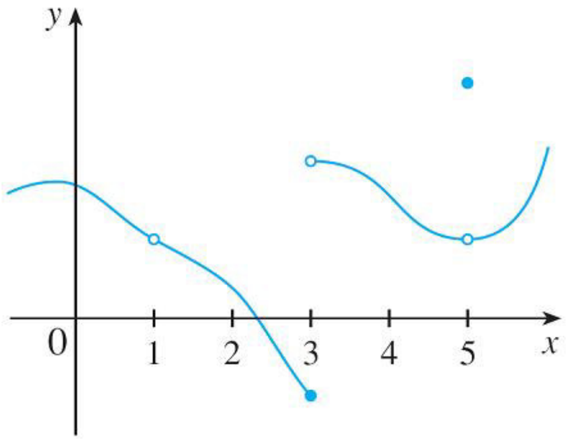
Continuity

- In fact, the change in $f(x)$ can be kept as small as we please by keeping the change in x sufficiently small

- If f is defined near a (in other words, f is defined on an open interval containing a, except perhaps at a), we say that f is **discontinuous at *a*** (or f has a **discontinuity** at a) if f is not continuous at a

Example

- We have the graph of a function f. At which values is f discontinuous? Why?



- Solution:
- It looks as if there is a discontinuity when $a = 1$ because the graph has a break there. The reason is that $f(1)$ is not defined

Example – *Solution*

cont'd

- The graph also has a break when $a = 3$, but the reason for the discontinuity is different. Here, $f(3)$ is defined, but $\lim_{x \to 3} f(x)$ does not exist (because the left and right limits are different). So f is discontinuous at 3

- What about $a = 5$? Here, $f(5)$ is defined and $\lim_{x \to 5} f(x)$ exists (because the left and right limits are the same)

- But $\lim_{x \to 5} f(x) \neq f(5)$

So f is discontinuous at 5

Example

- Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

- Solution:

- (a) Notice that *f*(2) is not defined, so *f* is discontinuous at 2. Later we'll see why *f* is continuous at all other values

Example – Solution

cont'd

- (b) Here $f(0) = 1$ is defined but $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x^2}$

  is not finite. So f is discontinuous at 0

- (c) Here $f(2) = 1$ is defined and

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} \quad = \lim_{x \to 2} \frac{(x - 2)(x + 1)}{x - 2}$$

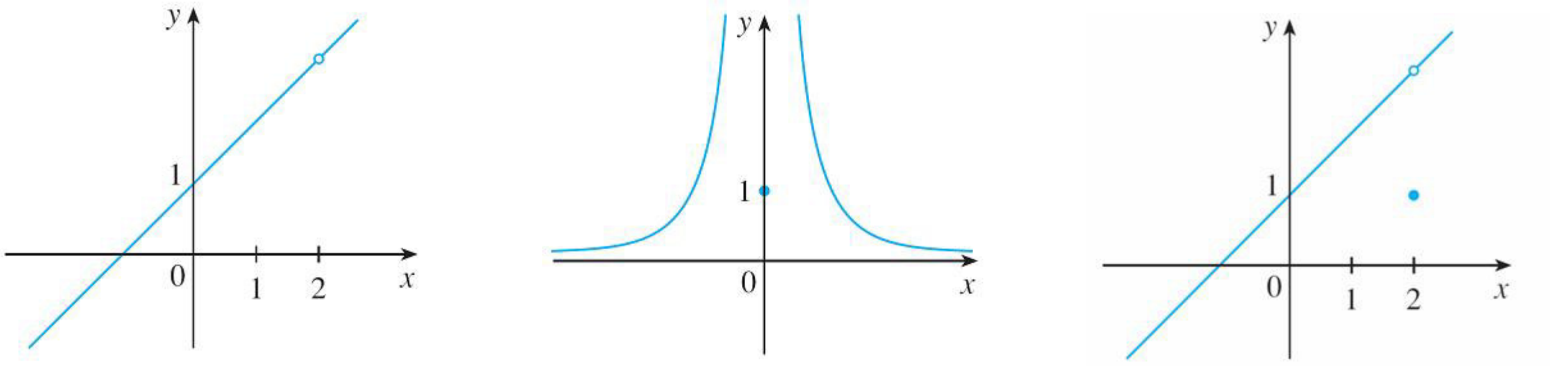
$$= \lim_{x \to 2} (x + 1) \quad = 3 \text{ exists}$$

- But $\lim_{x \to 2} f(x) \neq f(2)$

- so f is not continuous at 2

EXAMPLE – *SOLUTION*

- Here are the graphs of these functions



(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

Example – *Solution*

- In each case the graph can't be drawn without lifting the pen from the paper because a hole or break or jump occurs in the graph.

- The kind of discontinuity illustrated in parts (a) and (c) is called **removable** because we could remove the discontinuity by redefining f at just the single number 2. [The function $g(x) = x + 1$ is continuous]

- The discontinuity in part (b) is called an **infinite discontinuity**

Continuity

2 Definition A function f is **continuous from the right at a number *a*** if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at *a*** if

$$\lim_{x \to a^-} f(x) = f(a)$$

3 Definition A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left*.)

Continuity

- Instead of always using Definitions 1, 2, and 3 to verify the continuity of a function, it is often convenient to use the next theorem, which shows how to build up complicated continuous functions from simple ones

$\boxed{4}$ **Theorem** If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$ **2.** $f - g$ **3.** cf

4. fg **5.** $\dfrac{f}{g}$ if $g(a) \neq 0$

Continuity

5 Theorem

(a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

7 Theorem The following types of functions are continuous at every number in their domains:

polynomials	rational functions
root functions	trigonometric functions

8 Theorem If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

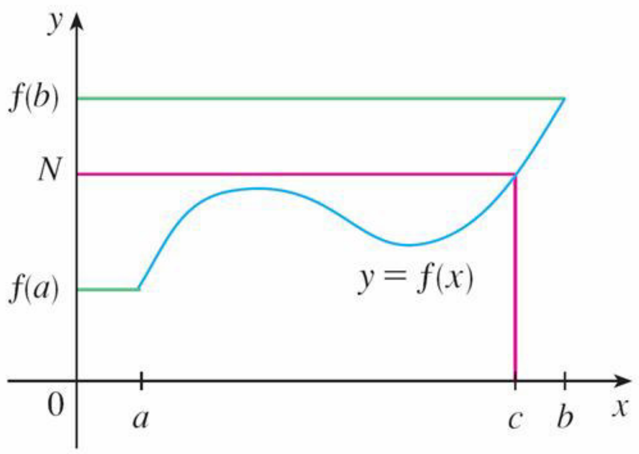
Continuity

9 Theorem If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

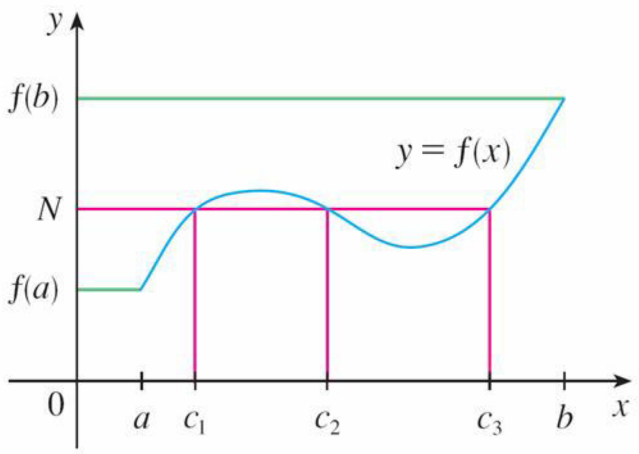
10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

Continuity

- The Intermediate Value Theorem states that a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$.
- Note that the value N can be taken on once [as in part (a)] or more than once [as in part (b)]



(a)

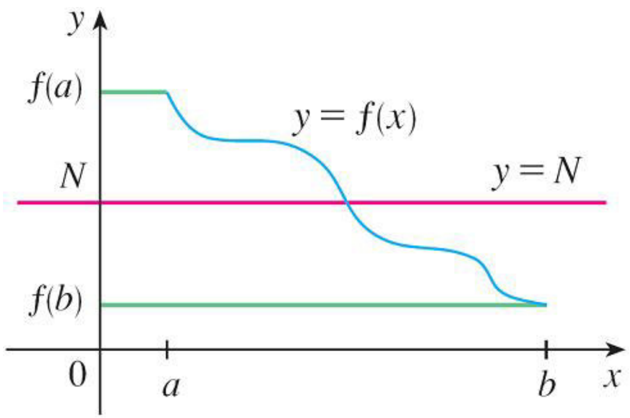


(b)

Continuity

- If we think of a continuous function as a function whose graph has no hole or break, then it is easy to see that the Intermediate Value Theorem is true

- In geometric terms it says that if any horizontal line $y = N$ is given between $y = f(a)$ and $y = f(b)$, then the graph of f can't jump over the line. It must intersect $y = N$ somewhere



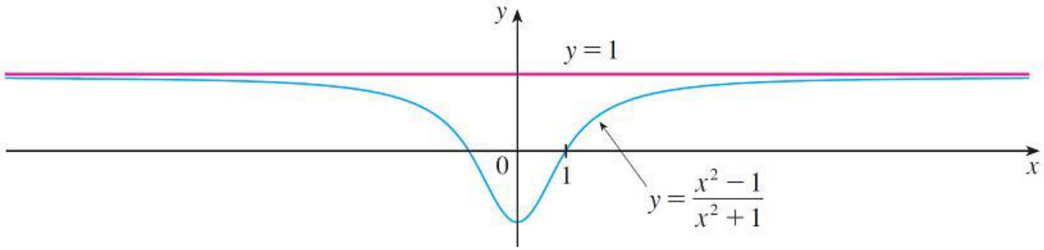
Continuity

- A simple consequence of the Intermediate Value theorem is the following Bolzano's Theorem
- Bolzano's theorem: Given a continuous function f such that $f(a)$ and $f(b)$ have opposite sign (i.e. either $f(a) < 0$ and $f(b) > 0$ or $f(a) > 0$ and $f(b) < 0$) then there is a point $c \in [a; b]$ such that $f(c) = 0$
- Proof. If $f(a)$ and $f(b)$ have opposite sign, then 0 is a value between $f(a)$ and $f(b)$ and so, by the Intermediate Value theorem, there exists a point $c \in [a; b]$ such that $f(c) = 0$

Horizontal Asymptotes

- Study the behavior of this f as x becomes large

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$



x	$f(x)$
0	−1
±1	0
±2	0.600000
±3	0.800000
±4	0.882353
±5	0.923077
±10	0.980198
±50	0.999200
±100	0.999800
±1000	0.999998

- As *x* grows larger *f*(*x*) gets closer to 1. We can make *f* (*x*) as close as we like to 1 by taking *x* sufficiently large

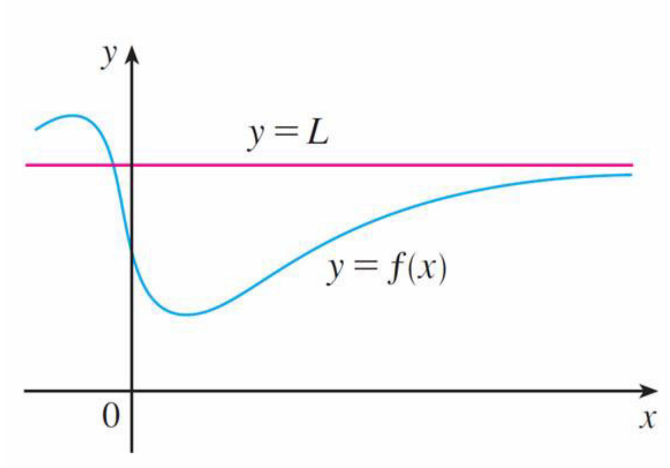
$$\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

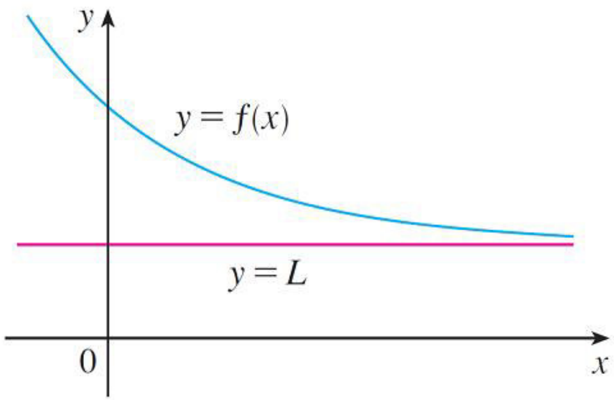
- So on both sides it tends to the horizontal line y=1

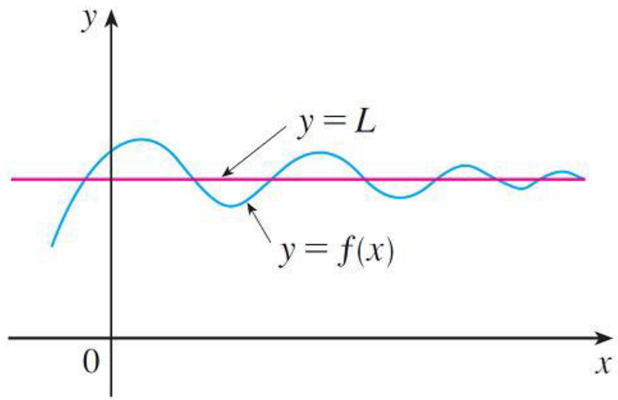
Horizontal Asymptotes

There are many ways for a graph to approach a line *y* = *L* (which is called a *horizontal asymptote*) as x increases

$$\lim_{x \to \infty} f(x) = L$$







Horizontal Asymptotes

- **3 Definition** The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$

Example

- Find $\lim_{x \to \infty} \frac{1}{x}$ and $\lim_{x \to -\infty} \frac{1}{x}$.

- Solution:

Observe that when x is large, 1/*x* is small. For instance,

$$\frac{1}{100} = 0.01 \qquad \frac{1}{10{,}000} = 0.0001 \qquad \frac{1}{1{,}000{,}000} = 0.000001$$

- In fact, by taking *x* large enough, we can make 1/*x* as close to 0 as we please

Example – *Solution*

cont'd

- Therefore, we have

- $$\lim_{x \to \infty} \frac{1}{x} = 0$$

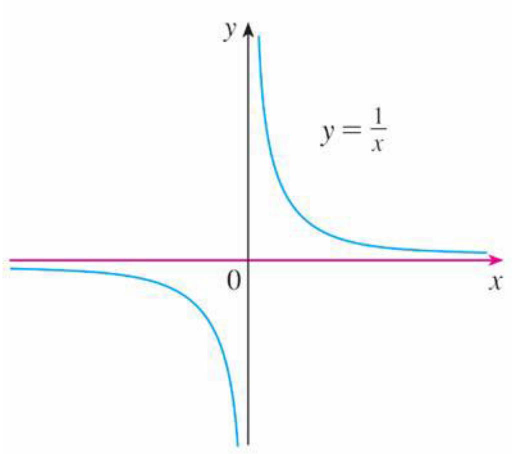
- Similar reasoning shows that when *x* is large negative, 1/*x* is small negative, so we also have

- $$\lim_{x \to -\infty} \frac{1}{x} = 0$$

Example – *Solution*

cont'd

- It follows that the line y = 0 (the x-axis) is a horizontal asymptote of the curve y = 1/x. (This is an equilateral hyperbola; also x=0 (the y-axis) is a vertical asymptote



$$\lim_{x \to \infty} \frac{1}{x} = 0, \quad \lim_{x \to -\infty} \frac{1}{x} = 0$$

Example 2

- Evaluate $\lim_{x \to \infty} \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$

- Solution:

As x becomes large, both numerator and denominator become large, so it isn't obvious what happens to their ratio. We need to do some preliminary algebra

- To evaluate the limit at infinity of any rational function, we first divide both the numerator and denominator by the highest power of x that occurs in the denominator. (We may assume that $x \neq 0$, since we are interested only in large values of x)

Example 2 – *Solution*

cont'd

- In this case the highest power of x in the denominator is x^2, so we have

$$\lim_{x\to\infty}\frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x\to\infty}\frac{\dfrac{3x^2 - x - 2}{x^2}}{\dfrac{5x^2 + 4x + 1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{3 - \dfrac{1}{x} - \dfrac{2}{x^2}}{5 + \dfrac{4}{x} + \dfrac{1}{x^2}}$$

Example 2 – *Solution*

cont'd

$$= \frac{\lim_{x \to \infty} \left(3 - \frac{1}{x} - \frac{2}{x^2}\right)}{\lim_{x \to \infty} \left(5 + \frac{4}{x} + \frac{1}{x^2}\right)}$$

(by Limit Law 5)

$$= \frac{\lim_{x \to \infty} 3 - \lim_{x \to \infty} \frac{1}{x} - 2 \lim_{x \to \infty} \frac{1}{x^2}}{\lim_{x \to \infty} 5 + 4 \lim_{x \to \infty} \frac{1}{x} + \lim_{x \to \infty} \frac{1}{x^2}}$$

(by 1, 2, and 3)

$$= \frac{3 - 0 - 0}{5 + 0 + 0}$$

(by 7)

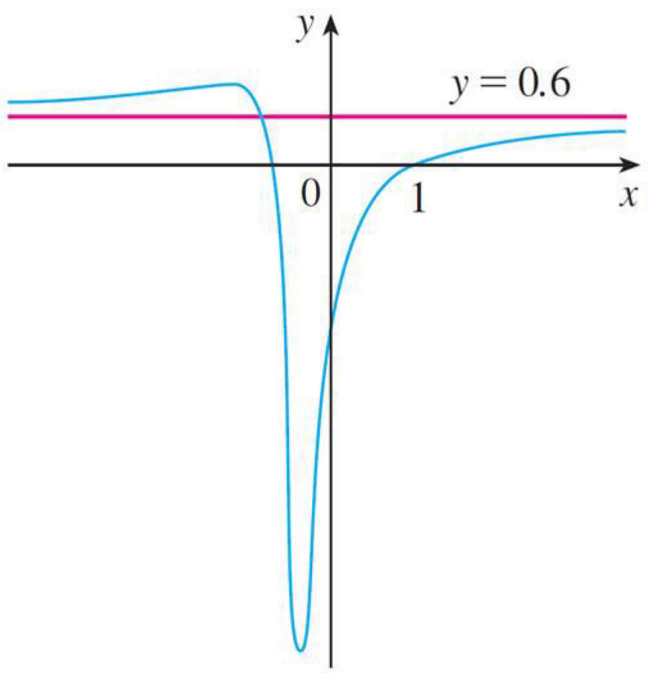
$$= \frac{3}{5}$$

Example 2 – *Solution*

cont'd

- A similar calculation shows that the limit as $x \to -\infty$ is also 3/5

- We can observe the graph of this function approaches the horizontal asymptote $y = 3/5$



$$y = \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

Example 3

- Find the horizontal and vertical asymptotes of the graph of the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

- Solution:

Dividing both numerator and denominator by *x* and using the properties of limits, we have

$$\lim_{x \to \infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \lim_{x \to \infty} \frac{\sqrt{2 + \frac{1}{x^2}}}{3 - \frac{5}{x}}$$

(since $\sqrt{x^2} = x$ for $x > 0$)

Example 3 – *Solution*

cont'd

$$= \frac{\lim_{x\to\infty} \sqrt{2 + \frac{1}{x^2}}}{\lim_{x\to\infty} \left(3 - \frac{5}{x}\right)}$$

$$= \frac{\sqrt{\lim_{x\to\infty} 2 + \lim_{x\to\infty} \frac{1}{x^2}}}{\lim_{x\to\infty} 3 - 5 \lim_{x\to\infty} \frac{1}{x}}$$

$$= \frac{\sqrt{2 + 0}}{3 - 5 \cdot 0}$$

Example 3 – *Solution*

cont'd

$$= \frac{\sqrt{2}}{3}$$

- Therefore the line $y = \sqrt{2}/3$ is a horizontal asymptote of the graph of f

- In computing the limit as $x \to -\infty$, we must remember that for $x < 0$, we have $\sqrt{x^2} = |x| = -x$

Example 3 – *Solution*

cont'd

- So when we divide the numerator by *x*, for $x < 0$ we get

$$\frac{1}{x}\sqrt{2x^2 + 1} = -\frac{1}{\sqrt{x^2}}\sqrt{2x^2 + 1}$$

$$= -\sqrt{2 + \frac{1}{x^2}}$$

Example 3 – *Solution*

cont'd

- Therefore

$$\lim_{x \to -\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \lim_{x \to -\infty} \frac{-\sqrt{2 + \dfrac{1}{x^2}}}{3 - \dfrac{5}{x}}$$

$$= \frac{-\sqrt{2 + \displaystyle\lim_{x \to -\infty} \frac{1}{x^2}}}{3 - 5 \displaystyle\lim_{x \to -\infty} \frac{1}{x}}$$

$$= -\frac{\sqrt{2}}{3}$$

Example 3 – *Solution*

cont'd

- Thus the line $y = -\sqrt{2}/3$ is also a horizontal asymptote
- A vertical asymptote is likely to occur when the denominator, $3x - 5$, is 0, that is, when $x = 5/3$

- If x is close to 5/3 and $x > 5/3$, then the denominator is close to 0 and $3x - 5$ is positive. The numerator $\sqrt{2x^2 + 1}$ is always positive, so $f(x)$ is positive.

- Therefore

$$\lim_{x \to (5/3)^+} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \infty$$

Example 3 – *Solution* cont'd

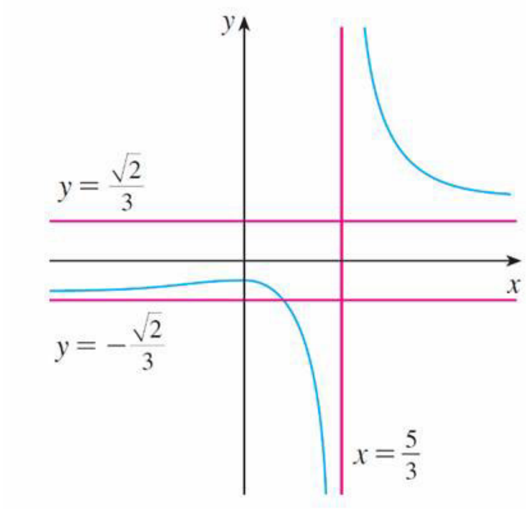
- If is close to 5/3 but $x < 5/3$, then $3x – 5 < 0$ and so $f(x)$ is large negative. Thus

$$\lim_{x \to (5/3)^-} \frac{\sqrt{2x^2 + 1}}{3x - 5} = -\infty$$

- The vertical asymptote is $x = 5/3$

So in the end we have

- three asymptotes



$$y = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$