On Exact and Approximate Stochastic Dominance Strategies for Portfolio Selection

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Abstract

One recent and promising strategy for Enhanced Indexation is the selection of portfolios that stochastically dominate the benchmark. We propose here a new type of approximate stochastic dominance rule which implies other existing approximate stochastic dominance rules. We then use it to find the portfolio that approximately stochastically dominates a given benchmark with the best possible approximation. Our model is initially formulated as a Linear Program with exponentially many constraints, and then reformulated in a more compact manner so that it can be very efficiently solved in practice. This reformulation also reveals an interesting financial interpretation. We compare our approach with several exact and approximate stochastic dominance models for portfolio selection. An extensive empirical analysis on real and publicly available datasets shows very good out-of-sample performances of our model.

Keywords: Applied probability, Stochastic dominance, Portfolio optimization, Expected shortfall, Index tracking

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1. Introduction

In this work we develop portfolio optimization methods for Enhanced Indexation (EI) based on various types of Stochastic Dominance (SD) criteria, and we compare their empirical performances. References on EI can be found in, e.g., Canakgoz and Beasley (2008); Guastaroba and Speranza (2012); Bruni et al. (2015). SD approaches to EI exhibit particular advantages and have an intuitive meaning in terms of Expected Utility Theory (see, e.g., Levy 1992, 2006). Furthermore, several relations between SD approaches and mean-risk optimization have been identified in the literature (see, e.g., Gotoh and Konno (2000) and references therein).

In most cases the optimization models for EI based on stochastic dominance have a large number of constraints, since a large number of conditions are needed to ensure SD. However, they can often be solved in reasonable time by taking advantage of polyhedral techniques developed in the field of Combinatorial Optimization. Ruszczyński and Vanderbei (2003) propose mean-risk models that are solvable by linear programming and generate portfolios whose returns are nondominated according to Second-order Stochastic Dominance (SSD). One of the first enhanced indexation models based on SD is also in Kuosmanen (2004). He derives and implements the first programs dealing with the exact First-order Stochastic Dominance (FSD) and SSD rules. Later, Luedtke (2008) describes compact linear programming formulations where the objective is to maximize the portfolio expected return with SSD constraints over the benchmark. An efficient practical approach to EI for large markets has been proposed by Fábián et al. (2011) and by Roman et al. (2013), who directly apply a SSD strategy to construct a portfolio whose return distribution dominates the one of a benchmark. More recently, Hodder et al. (2015) successfully apply the exact SSD methods of Kuosmanen (2004) and of Kopa and Post (2015), while Iñaki and Longarela (2015) provides a description of the set of all SSD-efficient portfolios by means of a family of mixed-integer linear constraints. Third-order Stochastic Dominance has also been recently applied to EI by Post and Kopa (2016).

As shown by Leshno and Levy (2002), relaxations of SD may provide advantages over exact SD in several economical contexts. Hence, they propose an approximate SD rule, called *Almost Stochastic Dominance*, and they identify the corresponding classes of utility functions for the case of first and second order stochastic dominance. An oversight in their work has been corrected in Tzeng et al. (2013), and further generalizations and characterizations have been provided in Levy et al. (2010); Tzeng et al. (2013); Post and Kopa (2013); Guo et al. (2013); Denuit et al. (2014), and Tsetlin et al. (2015). However, no applications of Almost Stochastic Dominance to portfolio selection seem to be available. This might be due to the difficulty of implementing Almost Stochastic Dominance rules in this setting, but also to the abundance of portfolios that typically dominate the benchmark already with standard SD rules.

Lizyayev and Ruszczynski (2012) have introduced a different relaxation of SD, which we call here *Lizyayev-Ruszczynski Almost Stochastic Dominance* (LR-ASD). In this case, the authors focus on computationally tractable conditions, and describe the optimization models for the practical implementation of first and second-order rules. They also describe potential applications of the LR-ASD rules to portfolio selection. However, they do not provide empirical results on real datasets, but only on some illustrative examples. Furthermore, also in this case one could question the advantage of a relaxed SD rule over the standard one which already guarantees an abundance of portfolios dominating the benchmark.

In contrast to the previous cases, under classical no-arbitrage assumptions, the existence of a portfolio dominating the benchmark is ruled out when using the standard Zero order (also called statewise) stochastic dominance. Thus, some kind of relaxed Zero order stochastic dominance is needed to find a portfolio dominating the benchmark. A preliminary study in this direction has been presented in Bruni et al. (2012), obtaining promising empirical and computational results on some real-world datasets.

We compare here several new and known variants of exact and approximate SD models for portfolio selection, and we analyze in detail their practical performances by means of an extensive comparative evaluation. Specifically, in Section 2 we briefly describe the main exact and approximate SD rules, and we define the Zero-order ε-Stochastic Dominance (ZεSD) rule, which implies both the Almost Stochastic Dominance rule introduced by Leshno and Levy (2002) and the one introduced by Lizyayev and Ruszczynski (2012). In Section 3 we present a cumulative version (CZεSD) of ZεSD and we apply it to the EI problem. The EI model based on CZεSD requires that the cumulative performance of the selected portfolio on all subsets of past observations outperforms that of the index up to an ε tolerance. This gives rise to a very large LP model which can however be reformulated in a compact manner and solved efficiently. Such reformulation also provides an interesting financial interpretation of the CZεSD approach to EI in terms of expected shortfall. In Section 4 we present empirical results on some major real world markets showing the practical effectiveness of several SD based approaches for portfolio selection and in particular of the one based on CZεSD.

To sum up, the main contributions of this work are the definition of new types of approximate stochastic dominance rules, their relations with the existing ones, and their application and interpretation in portfolio selection problems.

2. Exact and Approximate Stochastic Dominance Relations

According to Expected Utility Theory (see, e.g., von Neumann and Morgenstern 1944), a random variable is preferred to another if it presents a larger value of the expected utility. However, this approach depends on the specification of a utility function, which is a fairly subjective matter. On the other hand, Stochastic Dominance (SD), which is strictly related to Expected Utility Theory, is able to provide a (partial) order in the space of random variables avoiding the specification of a particular utility function, and for this reason it is particularly attractive to approach portfolio selection problems.

We now briefly recall the most common Stochastic Dominance order relations. Let A and B be two random variables, with distribution functions $F_A(\alpha) = \mathbf{Pr}(A \leq \alpha)$ and $F_B(\alpha) = \mathbf{Pr}(B \leq \alpha)$ for $\alpha \in \mathbb{R}$.

Definition 1. ***Zero-order Stochastic Dominance (ZSD):***
A is preferred to B w.r.t. ZSD if

$$F_{A-B}(0) = \mathbf{Pr}(A - B \leq 0) = 0. \tag{1}$$

In terms of the realizations a_t and b_t of A and B at time t, this means that $a_t \geq b_t$ almost everywhere.

Definition 2. ***First-order Stochastic Dominance (FSD):***
A is preferred to B w.r.t. FSD if

$$F_A(\alpha) \leq F_B(\alpha) \qquad \forall \alpha \in \mathbb{R}. \tag{2}$$

Definition 3. ***Second-order Stochastic Dominance (SSD):***
A is preferred to B w.r.t. SSD if

$$\int_{-\infty}^{\alpha} F_A(\tau) d\tau \ \leq \int_{-\infty}^{\alpha} F_B(\tau) d\tau \qquad \forall \alpha \in \mathbb{R}. \tag{3}$$

Note that, for the sake of simplicity, in the above definitions we omit the frequently added requirement for the strict inequality in at least one case. SD relations of any order v can be defined. When increasing the order, the corresponding condition becomes less restrictive: the v-th order SD implies the $(v+1)$-th order SD, while the opposite is not necessarily true (see, e.g., Levy 2006).

The ZSD relation represents behavior of a decision maker who prefers a random variable over another only when the first gives better outcomes than the second in (almost) all states of the world. On the other hand, higher order SD relations are less demanding and can be linked to Expected Utility Theory in terms of different classes of utility functions. Indeed, A is preferred to B w.r.t. FSD if and only if $E[u(A)] \geq E[u(B)]$ for all non-decreasing utility functions u; A is preferred to B w.r.t. SSD if and only if the same holds for all non-decreasing and concave utility functions (see, e.g., Levy 1992).

As showed, e.g., in Leshno and Levy (2002), there are cases where the above SD relations are not able to order the returns of two investments, even though most decision makers would prefer one investment over the other. Therefore, some relaxations of the above exact SD relations have been proposed in the literature with the aim of increasing their ability to establish preferences among investments. We first describe the one proposed by Leshno and Levy (2002) with the name of *Almost Stochastic Dominance*. This relationship can be specified for any order $v \geq 1$. With our notation, the one corresponding to the first order is:

Definition 4. ***Leshno-Levy Almost First-order Stochastic Dominance (LL-AFSD):***
Given a tolerance $\eta > 0$, A is preferred to B w.r.t. LL-AFSD if

$$\int_{S_1} (F_A(\tau) - F_B(\tau))d\tau \;\leq\; \eta \int_{\alpha'}^{\alpha''} |F_A(\tau) - F_B(\tau)|d\tau, \tag{4}$$

where $[\alpha', \alpha'']$ is the combined range of outcomes of A and B, and $S_1 = \{\tau \in [\alpha', \alpha''] : F_A(\tau) > F_B(\tau)\}$.

The underlying idea is to allow an area of possible violation of the classical SD, the so-called *actual violation area*, containing preferences of investors that can be considered *economically irrelevant*, as explained in detail in Leshno and Levy (2002). This corresponds to the exclusion of “extreme” utility functions and allows to fit in the theory situations where most of the investors would prefer investment A over investment B, but neither investment dominates the other with the usual FSD or SSD rules.

Another recent relaxation of Stochastic Dominance, still defined for any order $v \geq 1$, is proposed by Lizyayev and Ruszczynski (2012), who also provide the optimization models corresponding to First- and Second-order SD relations. However, the First-order relation requires, in this case, a large number of binary variables, so we focus on the more applicable Second-order condition.

Definition 5. ***Lizyayev-Ruszczynski Almost Second-order Stochastic Dominance (LR-ASSD):***
Given a tolerance $\vartheta > 0$, A is preferred to B w.r.t. LR-ASSD if

$$\int_{-\infty}^{\alpha} (F_A(\tau) - F_B(\tau))\, d\tau \;\leq \vartheta \qquad \forall \alpha \in [\alpha', \alpha''], \tag{5}$$

where $[\alpha', \alpha'']$ is the combined range of outcomes of A and B.

Note that SD relations of order higher than zero can lead to counterintuitive results, since increasing the order corresponds to neglecting some information. For example, even when A dominates B w.r.t. FSD, the difference $\mathbf{Pr}(B > A) - \mathbf{Pr}(A > B)$ can be arbitrarily close to 1 (Castagnoli 1983). On the other hand, the above difference is clearly not grater than 0 when A dominates B w.r.t. the Zero-order SD rule, which takes into account the full information of the random variables realizations. Therefore, if possible, in Enhanced Indexation one should aim for a portfolio whose in-sample return is preferred to the benchmark return w.r.t. ZSD. However, this condition cannot be fulfilled in practice, because otherwise arbitrage opportunities would exist (see, e.g., Meucci (2005); Bruni et al. (2013)). Hence, if we make the classical assumption of absence of arbitrage and we avoid higher order SD rules for the reasons mentioned above, then we should search for a portfolio dominating the benchmark w.r.t. some kind of *relaxed* ZSD. Thus, we propose the following new approximate SD relation.

Definition 6. ***Zero-order ε-Stochastic Dominance (ZεSD):***
Given a tolerance $\varepsilon > 0$, A is preferred to B w.r.t. ZεSD if

$$F_{A+\varepsilon-B}(0) = \mathbf{Pr}(A + \varepsilon - B \leq 0) = 0. \tag{6}$$

In terms of the realizations a_t and b_t at time t, this means that $a_t + \varepsilon \geq b_t$ almost everywhere. Even though our interest is in the zero order, this type of relation could be extended to higher orders.

Note that, in the case of approximate SD, large values of the tolerance (i.e., η, ϑ or ε, here generically denoted by λ) would cause indifference, in the sense that a variable A dominates B and, *at the same time*, B dominates A. However, let us denote by

$$\lambda(A, B) \; = \; \inf\{\lambda : A \text{ is preferred to } B \text{ with tolerance } \lambda\}$$

w.r.t. the approximate SD under analysis. Now, if $\lambda(A, B) > \lambda(B, A)$, one will prefer B to A, and vice versa if $\lambda(A, B) < \lambda(B, A)$. This allows to order all pairs of random variables but those for which we have the unlikely equality $\lambda(A, B) = \lambda(B, A)$.

We now relate the ZεSD condition to other approximate first and second order stochastic dominance rules. More specifically, straightforward arguments can be used to show that ZεSD implies both LL-AFSD and LR-ASSD with appropriate tolerances.

Remark 7 (ZεSD vs. LL-AFSD). *Let A and B be two random variables with absolutely continuous distribution functions F_A and F_B, respectively. If A is preferred to B w.r.t. ZεSD, then A is preferred to B w.r.t. LL-AFSD with tolerance $\eta = \left(\int_{\alpha'}^{\alpha''} |F_A(\tau) - F_B(\tau)| d\tau\right)^{-1} \varepsilon$.*

Note that the tolerance η in LL-AFSD is an upper bound on the ratio of the area of violation of FSD (where F_A is above F_B) and the total area enclosed between F_A and F_B. When A is preferred to B according to ZεSD, the area of violation of FSD is bounded by ε, so that LL-AFSD follows for all η not smaller than the ratio between ε and the total area enclosed between F_A and F_B.
As a consequence of Remark 7 and of the results in Leshno and Levy (2002), if A is preferred to B according to ZεSD, then $E[u(A)] \geq E[u(B)]$ for all utility functions u in the set $U_1^*(\eta)$ described in Leshno and Levy (2002) for $\eta = \left(\int_{\alpha'}^{\alpha''} |F_A(\tau) - F_B(\tau)| d\tau\right)^{-1} \varepsilon$.

Remark 8 (ZεSD vs. LR-ASSD). *Let A and B be two random variables with absolutely continuous distribution functions F_A and F_B, respectively. If A is preferred to B w.r.t. ZεSD, then A is preferred to B w.r.t. LR-ASSD with tolerance $\vartheta = \varepsilon$.*

As a consequence of Remark 8 and of the results in Lizyayev and Ruszczynski (2012), if A is preferred to B according to ZεSD, then $E[u(A)] + \varepsilon \geq E[u(B)]$ for any nondecreasing concave utility function u with first derivative $u' \leq 1$.

3. Approximate Stochastic Dominance for EI

EI models are built using the price values of n assets and of the benchmark index I over a time interval. We use the following notation:

r_t^I is the benchmark index return at time $t = 1, \dots, m$;

r_{it} is the return of asset i at time t for $i = 1, \dots, n$ and $t = 1, \dots, m$;

x is the EI portfolio we are selecting. Its components x_i are the fractions of the given capital invested in asset i for that portfolio.

$R_t(x) = \sum_{i=1}^{n} x_i r_{it}$ is the portfolio return at time $t = 1, \dots, m$;

$\delta_t(x) = R_t(x) - r_t^I$ is the *excess return*, or overperformance, of the selected portfolio w.r.t. the benchmark index at time $t = 1, \dots, m$.

A portfolio x having $\delta_t(x) < 0$ underperforms the benchmark index at time t, while a portfolio with $\delta_t(x) > 0$ overperforms it.

Portfolio returns $R_t(x)$ and benchmark returns r_t^I can be considered as the realizations, for each time t, of two random variables, called Portfolio Return (PR) and Benchmark Return (BR), respectively. The historical excess return $\delta_t(x)$ may be considered as the equally likely t-th realization of the difference between the discrete random variables PR and BR. Let T denote the set of in-sample time periods. By Definition 6, PR is preferred to BR w.r.t. ZεSD if

$$\delta_t(x) \geq -\varepsilon \qquad \forall t \in T.$$

This means that the excess return $\delta_t(x)$ can be negative for some of the in-sample time periods (i.e., an underperformance w.r.t. the benchmark), but in any case it cannot be smaller than $-\varepsilon$ (the underperformance is limited). In other words, requiring ZεSD over the benchmark provides, in each period, a bound on the possible loss w.r.t. the benchmark. However, even though the loss in each period is small, the cumulative loss in all periods could still be large, if many losses occur.

To bound such cumulative losses, we introduce a cumulative version of Zero-order ε-Stochastic Dominance which: (i) implies the ordinary ZεSD; (ii) has an interesting financial interpretation in terms of expected shortfall (see Section 3.1); (iii) seems to provide good out-of-sample performance (see Section 4.2).

Definition 9. ***Cumulative Zero-order ε-Stochastic Dominance (CZεSD):*** *Given a tolerance $\varepsilon > 0$, A is preferred to B w.r.t. CZεSD if*

$$\sum_{t \in S} a_t + \varepsilon \geq \sum_{t \in S} b_t \qquad \forall S \subseteq T, \tag{7}$$

where a_t and b_t are the realizations of A and B for all t in T.

Thus, in the EI case, we have that PR is preferred to BR w.r.t. CZεSD if

$$\sum_{t \in S} \delta_t(x) \geq -\varepsilon \quad \forall S \subseteq T. \tag{8}$$

Condition (8) means that $\delta_t(x)$ can be negative for some subsets of the in-sample window (i.e., a cumulative underperformance w.r.t. the benchmark), but in any case the value of the above sum cannot be smaller than $-\varepsilon$ (the cumulative underperformance is limited).

Clearly, if ε is fixed *a priori*, a portfolio dominating the index might not exist. However, we can search for the smallest value of ε for which such a portfolio exists, as described in the following section. It is easy to see that CZεSD implies ZεSD: a limited cumulative underperformance implies that the underperformance for any time period is also limited. However, when ε tends to zero, both rules collapse to ZSD, which is theoretically prevented by the no-arbitrage argument. We also remark that, for a given value of ε, it may happen that portfolio P_1 dominates portfolio P_2 and, *at the same time*, portfolio P_2 dominates portfolio P_1 w.r.t. (C)ZεSD. However, as observed after Definition 6 in Section 2, this problem is typically removed by minimizing ε, which is the aim of the following section.

3.1. Optimization Models for CZεSD

Among all portfolios that are preferred to the benchmark index with respect to the CZεSD criterion, we are interested in the one(s) having the smallest value for ε. This can be obtained by solving an optimization problem. The above stochastic dominance conditions can be formulated as constraints that we call *limiting* constraints. As usual, we also require the budget constraint ($\sum_{i=1}^{n} x_i = 1$), the no short-selling condition ($x_i \geq 0 \quad \forall i$), and we allow for the possibility of a set C of other linear constraints, such as the request that the portfolio expected return is greater than or equal to a target return level. We thus obtain the following Linear Programming problem that minimizes the greatest underperformance ε by maximizing $-\varepsilon$:

$$\begin{cases} \max & -\varepsilon \\ \text{s.t.} & \displaystyle\sum_{t \in S} \delta_t(x) \geq -\varepsilon \quad \forall S \subseteq T \\ & \displaystyle\sum_{i=1}^{n} x_i = 1 \\ & x \in C \\ & x \in \mathbb{R}^n_+ \\ & \varepsilon \in \mathbb{R}. \end{cases} \tag{9}$$

Note that the number of limiting constraints is exponential in m: one for every subset S of T. Since typical values for m may range between 100 and 500, the number of constraints may be huge. Nevertheless, we observe that, using the equivalence of optimization and separation established in Grötschel et al. (1993), Problem (9) can be theoretically solved in polynomial time since, for any $x \in \mathbb{R}^n_+$, we can efficiently solve the following *separation problem*: find a set of time periods $V \subseteq T$ such that

$$\sum_{t \in V} \delta_t(x) < -\varepsilon, \tag{10}$$

or conclude that no such set exists. In this case, the separation problem can be solved by simply checking if the set $\{t \in T : \delta_t(x^\star) < 0\}$ satisfies (10) or not, so (9) is solvable for instance by a constraint generation approach.

However, even better, we now show that Problem (9) can be efficiently solved, both in theory and in practice, by reformulating it as a Linear Program with a polynomial number of constraints, which also has an interesting financial interpretation. Indeed, Problem (9) can be written as

$$\begin{cases} \max\limits_{x \in \mathbb{R}^n_+} & \min\limits_{S \subseteq T} \delta_S(x) \\ \text{s.t.} & \sum\limits_{i=1}^{n} x_i = 1 \\ & x \in C \end{cases} \tag{11}$$

where $\delta_S(x) = \sum_{t \in S} \delta_t(x)$ for $S \subseteq T$. Note that

$$\min_{S \subseteq T} \delta_S(x) = \sum_{t \in T} \min\{0, \delta_t(x)\} = -\sum_{t \in T} \max\{0, -\delta_t(x)\}.$$

Since

$$\max_{x \in \mathbb{R}^n_+} \min_{S \subseteq T} \delta_S(x) = -\min_{x \in \mathbb{R}^n_+} \sum_{t \in T} \max\{0, -\delta_t(x)\},$$

the CZεSD model is equivalent to minimizing the expected shortfall of the portfolio below the benchmark. Furthermore, we can linearize Problem (11) with auxiliary variables y_t, for $t \in T$, in a classical manner

$$\begin{cases} \min & \sum\limits_{t \in T} y_t \\ \text{s.t.} & \\ & y_t + \delta_t(x) \geq 0 \qquad\qquad t \in T \\ & \sum\limits_{i=1}^{n} x_i = 1 \\ & x \in C \\ & x \in \mathbb{R}^n_+, \quad y \in \mathbb{R}^{|T|}_+ \end{cases} \tag{12}$$

Note that Problem (12) has only $n + |T|$ variables and $|T| + 1$ constraints in addition to those defining C, and can thus be solved very efficiently in practice even for large markets and extensive in-sample periods.

4. Empirical Analysis

This section presents an extensive empirical analysis of some models that, among the various SD-based approaches proposed in the literature for portfolio selection, appear the most promising and practically realizable ones. In addition, we also consider the pure Mean-Variance (MV) approach (Markowitz 1959), usually regarded as the benchmark model for asset allocation. We thus compare the six portfolio selection models listed below.

- **CZεSD**: the portfolio having CZεSD w.r.t. the market index, obtained by solving (12) as described in Section 3.1;
- **RMZ-SSD**: the portfolio having SSD w.r.t. the market index, obtained by implementing cutting planes techniques as explained in Roman et al. (2013);
- **LR-ASSD**: the portfolio having LR-ASSD w.r.t. the market index, obtained by implementing model (12) of Lizyayev and Ruszczynski (2012) specialized to the case of portfolio selection, i.e., assuming equal probabilities $\pi_i = 1/m$ for $i = 1, \dots, m$ and vector $x \geq 0$ such that $\sum_k x_k = 1$;
- **L-SSD**: the portfolio having L-SSD w.r.t. the market index, obtained by implementing model (cSSD1) of page 1438 of Luedtke (2008);
- **KP-SSD**: the portfolio having SSD w.r.t. the market index, obtained by implementing model (10) of Kopa and Post (2015), where the weight vector is fixed as in (6) of Hodder et al. (2015) with $\gamma = 3$. The authors called the resulting portfolio KP2011Power3. This has generally better out-of-sample performance than the model in Kuosmanen (2004) or other variants in Kopa and Post (2015) (named KP2011Min, KP2011Av);
- **MeanVar**: the reference Mean-Variance portfolio, as introduced in Markowitz (1959).

4.1. Data Sets

We test all the above strategies on several real-world datasets belonging to major stock markets across the world. We first provide detailed results on the following:

1. **DJIA** (Dow Jones Industrial Average, USA), containing 28 assets and 1363 observations (February 1990 - April 2016);
2. **NASDAQ 100** (National Association of Securities Dealers Automated Quotation, USA), containing 82 assets and 596 observations (November 2004 - April 2016);
3. **FTSE 100** (Financial Times Stock Exchange, UK), containing 83 assets and 717 observations (July 2002 - April 2016);
4. **SP500** (Standard & Poor's, USA), containing 442 assets and 595 observations (November 2004 - April 2016).

5. **FF49** (Fama & French 49 Industry portfolios, USA), containing 49 portfolios considered as assets (using the subsample where all the returns of the 49 industries are available, namely from July 1969 to July 2015);

The first 4 datasets consist of weekly linear returns computed on daily prices data, adjusted for dividends and stock splits, obtained from *Thomson Reuters Datastream*. We included stocks with at least ten years of observations. Furthermore, when necessary, the assets prices are filtered to check and to correct inaccurate data. We use the market index as benchmark. The corresponding weekly returns time series for assets and indexes are publicly available in Bruni et al. (2016) for research purposes.

The last dataset was obtained from `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`. In this case, we converted daily into weekly returns, thus having 2325 observations, and we used as benchmark index the Equally-Weighted portfolio.

In our analysis, we adopt a Rolling Time Window (RTW) scheme of evaluation: we allow for the possibility of rebalancing the portfolio composition during the holding period, at fixed intervals. A key point of the RTW scheme concerns the calibration of the in-sample and of the out-of-sample periods. On the basis of the findings of Jegadeesh and Titman (2001) and of a preliminary empirical analysis, we chose to adopt a period of 52 weeks for the in-sample window and of 12 weeks for the out-of-sample window, with rebalancing allowed every 12 weeks.

4.2. Performance Measures and Results

In Portfolio Optimization, the out-of-sample performance of a portfolio is generally evaluated by using a number of *performance measures*. For our analysis we choose the following five performance measures typically adopted in the literature (see, e.g., Rachev et al. 2008; DeMiguel et al. 2009, and references therein). We denote by R^{out} the out-of-sample portfolio return, by R_I^{out} the index return in the out-of-sample period, and by r_f a constant risk free rate of return that we set equal to 0.

- **Sharpe Ratio** (Sharpe 1966, 1994) is defined as the ratio between the average of $R^{out} - r_f$ and its standard deviation, namely:

$$\frac{E[R^{out} - r_f]}{\sigma(R^{out})}.$$

  The larger is its value, the better is the portfolio performance particularly w.r.t. the central part of the portfolio return distribution.

- **Sortino Ratio** (Sortino and Satchell 2001), defined as the ratio between the average of $R^{out} - r_f$ and its downside deviation, namely:

$$\frac{E[R^{out} - r_f]}{\sigma(\min\{R^{out} - r_f,\, 0\})}.$$

  The larger is its value, the better is the portfolio performance.

- **Rachev Ratio** (Rachev et al. 2004), defined as the ratio between the average of the best $\beta\%$ values of $R^{out} - r_f$ (with the opposite sign) and that of the worst $\alpha\%$ values of $R^{out} - r_f$. More precisely, the Rachev Ratio is based on the notion of *Conditional Value-at-Risk* at a specified confidence level α, $CVaR_\alpha$ (see, e.g., Rockafellar and Uryasev 2000), and has the following formulation:

$$\frac{CVaR_\beta(r_f - R^{out})}{CVaR_\alpha(R^{out} - r_f)}.$$

The larger is its value, the better is the portfolio performance particularly w.r.t. the tails of the portfolio return distribution. Parameters α and β have been set at 5%.

- **Information Ratio** (Goodwin 1998), defined as the expected value of the difference between the out-of-sample portfolio return and that of the benchmark index divided by the standard deviation of such difference, namely:

$$\frac{E[R^{out} - R_I^{out}]}{\sigma(R^{out} - R_I^{out})}.$$

The larger is its value, the better is the portfolio performance. This measure is particularly used by practitioners because it is a kind of "signal-to-noise ratio" for a portfolio manager.

- **Turnover**, defined as the average on all rebalances of the sum of the absolute values of the trades among the n available assets, namely:

$$\frac{1}{N_{reb}} \sum_{j=1}^{N_{reb}} \sum_{i=1}^{n} |x_{j,i} - x_{j-1,i}|,$$

where N_{reb} is the number of rebalances and $x_{j,i}$ is the weight of asset i for the j-th rebalance (see DeMiguel et al. 2009). This measure is often used to approximately capture transaction costs and takes into account only the amount of trading generated by the model at each rebalance without considering the changes due to variations in asset prices.

- **Jensen's Alpha** (Jensen 1968), defined as the intercept of the line given by the linear regression of $R^{out} - r_f$ on $R_I^{out} - r_f$, namely:

$$\alpha = (E[R^{out}] - r_f) - \beta(E[R_I^{out}] - r_f),$$

where $\beta = Cov(R^{out}, R_I^{out})/\sigma^2(R_I^{out})$ is the regression coefficient representing the systematic risk of the selected portfolio w.r.t. the market.

- **Average Return**, defined as the average out-of-sample return $E[R^{out}]$ of a portfolio.

Table 1: Out-of-sample results for DJIA

	DJIA						
Approach	Sharpe	Sortino	Rachev	Info R.	Turnover	Jensen's	Aver. Return
CZεSD	0.09	0.14	**1.10**	**0.09**	**0.63**	0.0006	0.0023
RMZ-SSD	0.07	0.10	0.98	0.00	0.91	0.0004	0.0017
LR-ASSD	0.08	0.11	**1.10**	0.02	1.12	0.0005	0.0020
L-SSD	0.08	0.11	1.01	0.01	0.98	0.0004	0.0017
KP-SSD	**0.12**	**0.17**	1.02	0.08	1.07	**0.0023**	**0.0036**
MeanVar	0.09	0.12	0.99	0.01	0.71	0.0006	0.0018
Index	0.07	0.09	1.02	-	-	-	0.0016

Table 2: Out-of-sample results for NASDAQ100

	NASDAQ100						
Approach	Sharpe	Sortino	Rachev	Info R.	Turnover	Jensen's	Aver. Return
CZεSD	0.12	0.17	1.03	**0.14**	1.21	0.0016	0.0040
RMZ-SSD	0.10	0.14	0.94	0.02	1.19	0.0011	0.0028
LR-ASSD	0.09	0.13	1.00	0.02	1.31	0.0009	0.0026
L-SSD	**0.13**	**0.20**	**1.08**	0.07	1.37	0.0017	0.0034
KP-SSD	0.12	0.16	1.02	0.09	1.26	**0.0033**	**0.0055**
MeanVar	0.11	0.17	1.08	0.02	**0.92**	0.0012	0.0027
Index	0.08	0.10	1.02	-	-	-	0.0023

Table 3: Out-of-sample results for FTSE100

	FTSE100						
Approach	Sharpe	Sortino	Rachev	Info R.	Turnover	Jensen's	Aver. Return
CZεSD	0.09	0.12	0.93	**0.18**	1.23	0.0014	0.0023
RMZ-SSD	**0.15**	**0.20**	0.99	0.15	1.02	0.0026	0.0032
LR-ASSD	0.07	0.10	0.95	0.05	1.36	0.0011	0.0018
L-SSD	0.13	0.17	1.01	0.14	1.41	0.0022	0.0028
KP-SSD	0.14	**0.20**	**1.13**	0.14	1.13	**0.0042**	**0.0050**
MeanVar	0.13	0.17	1.00	0.13	**0.94**	0.0021	0.0027
Index	0.04	0.05	0.91	-	-	-	0.0009

Table 4: Out-of-sample results for SP500

	SP500						
Approach	Sharpe	Sortino	Rachev	Info R.	Turnover	Jensen's	Aver. Return
CZεSD	0.08	0.11	0.97	**0.10**	1.70	0.0012	0.0026
RMZ-SSD	**0.09**	**0.12**	0.93	0.04	1.30	0.0012	0.0021
LR-ASSD	**0.09**	**0.12**	**1.12**	0.07	1.67	0.0009	0.0019
L-SSD	0.08	0.10	0.98	0.03	1.54	0.0003	0.0013
KP-SSD	0.08	0.11	1.01	0.06	1.42	**0.0022**	**0.0036**
MeanVar	**0.09**	0.11	0.98	0.03	**1.12**	0.0009	0.0018
Index	0.05	0.06	1.03	-	-	-	0.0013

Table 5: Out-of-sample results for FF49

	FF49						
Approach	Sharpe	Sortino	Rachev	Info R.	Turnover	Jensen's	Aver. Return
CZεSD	0.20	0.25	1.04	**0.15**	1.08	0.0007	0.0050
RMZ-SSD	0.22	**0.29**	**1.13**	0.02	0.93	0.0015	0.0045
LR-ASSD	0.14	0.18	1.04	-0.12	1.21	-0.0004	0.0030
L-SSD	0.22	0.28	1.07	-0.01	1.14	0.0012	0.0042
KP-SSD	0.20	0.26	1.05	0.08	1.07	**0.0023**	**0.0060**
MeanVar	**0.23**	0.28	1.08	-0.02	**0.74**	0.0013	0.0041
Index	0.17	0.21	1.03	-	-	-	0.0043

All the procedures have been implemented in MATLAB 8.0 and executed on a workstation with Intel Core2 Duo CPU (T7500, 2.2 GHz, 4Gb RAM) under MS Windows Vista. The linear and quadratic programming problems have been solved using the TOMLAB/CPLEX toolbox (Holmstrom et al. 2012). Running times for CZεSD, which requires only the solution of one medium size LP, is always within one second. The other approaches consist in solving larger LPs or several LPs, and their solution typically requires some minutes. Hence, from a computational point of view, CZεSD is clearly preferable for large markets and wide in-sample windows.

For each dataset and for each portfolio strategy we provide the out-of-sample performance results in Tables 1-5, where the best results are marked in bold. All the portfolios generated in our experiments are available in Bruni et al. (2016). First, we note that the described models based on SD appear suitable for EI, since the obtained portfolios typically outperform the market index. Furthermore, we observe that for the Sharpe and Sortino ratios there is no clear dominance among the different models, but rather a pool of approaches providing good performances (CZεSD, RMZ-SSD, KP-SSD, MeanVar). Evidently, none of them is able to completely outperform the others when focusing on the central part of the portfolio return distribution. A similar situation arises for the Rachev ratio, referring to the tails of the return distribution, but the composition of the pool is different (LR-ASSD, L-SSD, KP-SSD). On the other hand, for the Information Ratio, a noteworthy performance measure among practitioners, CZεSD is clearly better than all other approaches. Thus, it seems that CZεSD, i.e., the minimization of the expected shortfall of the portfolio below the benchmark, is able to provide more persistent and less volatile excess returns than the other methods, although some other approaches show better properties for the tails of the return distributions. We stress that good results for the Information Ratio are not only fulfilling the theoretical aims of EI but are generally considered highly desirable in real applications. We then observe that the MeanVar model has the best result in terms of the portfolio turnover, while the KP-SSD has the best result for Jensen's Alpha. Finally, it appears that the best average returns are provided by KP-SSD, immediately followed by CZεSD.

To provide further evidence of the robustness of our approach, we report aggregate results for the same performance measures on 16 publicly available datasets. In addition to the previous 5 datasets, we consider the 8 datasets from Beasley's OR-Library (Hang Seng 31, DAX 100, FTSE 100, S&P 100, Nikkei 225, S&P 500, Russell 2000, Russell 3000, available from `http://people.brunel.ac.uk/~mastjjb/jeb/orlib/indtrackinfo.html`) and 3 datasets that have been used in Cesarone et al. (2013, 2015) (EuroStoxx 50, MIBTEL 230, NASDAQ 2200, available from `http://host.uniroma3.it/docenti/cesarone/DataSets.html`).

For each of these 16 dataset, we compute the ranking of the models for each of the above performance measures. Then, for each model, we compute the median of those values across all datasets, as shown in Table 6. This should give an evaluation of the typical performance of each model according to each different performance measure. We also report, in the same table, an indicator of the "overall performance" of each model, computed as the median of all its medians (hence smaller values are better). According to this indicator, the CZεSD, RMZ-SSD and KP-SSD models provide the best out-of-sample performance. Of course this overall performance is a simplification, but it is intended to give a concise (even though approximate) answer to the very basic question "how effective is each model?".

Table 6: Summary of results on 16 datasets

Approach	Sharpe	Sortino	Rachev	Info R.	Turnover	Jensen's	Aver. Return	Overall Perf.
CZεSD	**2**	3	5	**1**	4	3	2	**3**
RMZ-SSD	3	**2**	4	3	2	3	3	**3**
LR-ASSD	6	5	4	5	6	5	5	5
L-SSD	4	4	5	5	5	5	5	5
KP-SSD	4	3	**3**	3	3	**1**	**1**	**3**
MeanVar	3	3	4	4	**1**	4	5	4
Index	7	7	7	-	-	-	7	7

To improve the practical performance of the selected portfolios, we briefly describe some additional features that can be integrated in SD-based models without computational overload. A first variant can be applied in the CZεSD and RMZ-SSD models, and consists in introducing linear constraints requiring that the in-sample expected return $\mu(T) = \frac{1}{|T|}\sum_{t\in T} R_t(x)$ is not smaller than a threshold proportional to the expected return $\mu^{EW}(T)$ of the Equally-Weighted (EW) portfolio, i.e., the portfolio where the capital is equally distributed among all assets. This is an easily computable portfolio that in average practical cases exhibits fair performances.The second variant tries to complement SD strategies with the low-variance advantages given by the Mean-Variance approach. This is realized by restricting the various SD models to use only those assets included in the Mean-Variance portfolio by means of simple linear constraints. The investigation of these lines of research will be the object of future work.

5. Conclusions

Stochastic dominance approaches to the Enhanced Indexation problem seem to be very attractive from a theoretical viewpoint. However, some issues need to be addressed for their practical application. First, exact stochastic dominance may often fail to order a given pair of random variables. Second, the lowest order exact stochastic dominance relations conflict with the classical no-arbitrage conditions in financial markets. Finally, the stochastic dominance models proposed in the literature are often too large to be solved in real-world markets. In this work we deal with all these issues and propose a new approximate stochastic dominance rule. This rule admits a financial interpretation in terms of expected shortfall, which also leads to a linear programming formulation that can be efficiently solved. A comprehensive empirical analysis shows the good practical performance of the proposed model.

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