First Order Logic & Conjunctive Queries

Formal Methods

Giuseppe De Giacomo

Sapienza Università di Roma
MSc in Engineering in Computer Science

First-order logic

- First-order logic (FOL) is the logic to speak about objects, which are the domain of discourse or universe.
- FOL is concerned about properties of these objects and relations over objects (resp., unary and n-ary predicates).
- FOL also has functions including constants that denote objects.

FOL syntax – Terms

We first introduce:

- A set $Vars = \{x_1, \ldots, x_n\}$ of individual variables (i.e., variables that denote single objects).
- A set of functions symbols, each of given arity ≥ 0.
  Functions of arity 0 are called constants.

Def.: The set of *Terms* is defined inductively as follows:

- $Vars \subseteq Terms$;
- If $t_1, \ldots, t_k \in Terms$ and f^k is a k-ary function symbol, then $f^k(t_1, \ldots, t_k) \in Terms$;
- Nothing else is in *Terms*.

FOL syntax – Formulas

Def.: The set of *Formulas* is defined inductively as follows:

- If $t_1, \ldots, t_k \in Terms$ and P^k is a k-ary predicate, then $P^k(t_1, \ldots, t_k) \in Formulas$ (atomic formulas).
- If $t_1, t_2 \in Terms$, then $t_1 = t_2 \in Formulas$.
- If $\varphi \in Formulas$ and $\psi \in Formulas$ then
  - $\neg\varphi \in Formulas$
  - $\varphi \wedge \psi \in Formulas$
  - $\varphi \vee \psi \in Formulas$
  - $\varphi \rightarrow \psi \in Formulas$
- If $\varphi \in Formulas$ and $x \in Vars$ then
  - $\exists x.\varphi \in Formulas$
  - $\forall x.\varphi \in Formulas$
- Nothing else is in *Formulas*.

Note: a predicate of arity 0 is a proposition of propositional logic.

Interpretations

Given an alphabet of predicates $P_1, P_2, \ldots$ and functions $f_1, f_2, \ldots$, each with an associated arity, a FOL interpretation is:

$$\mathcal{I} = (\Delta^{\mathcal{I}}, P_1^{\mathcal{I}}, P_2^{\mathcal{I}}, \ldots, f_1^{\mathcal{I}}, f_2^{\mathcal{I}}, \ldots)$$

where:

- $\Delta^{\mathcal{I}}$ is the domain (a set of objects)
- if P_i is a k-ary predicate, then $P_i^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \cdots \times \Delta^{\mathcal{I}}$ (k times)
- if f_i is a k-ary function, then $f_i^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \cdots \times \Delta^{\mathcal{I}} \longrightarrow \Delta^{\mathcal{I}}$ (k times)
- if f_i is a constant (i.e., a 0-ary function), then $f_i^{\mathcal{I}} : () \longrightarrow \Delta^{\mathcal{I}}$ (i.e., f_i denotes exactly one object of the domain)

Assignment

Let *Vars* be a set of (individual) variables.

Def.: Given an interpretation $\mathcal{I}$, an assignment is a function

$$\alpha : \mathit{Vars} \longrightarrow \Delta^{\mathcal{I}}$$

that assigns to each variable $x \in \mathit{Vars}$ an object $\alpha(x) \in \Delta^{\mathcal{I}}$.

It is convenient to extend the notion of assignment to terms. We can do so by defining a function $\hat{\alpha} : \mathit{Terms} \longrightarrow \Delta^{\mathcal{I}}$ inductively as follows:

- $\hat{\alpha}(x) = \alpha(x)$, if $x \in \mathit{Vars}$
- $\hat{\alpha}(f(t_1, \ldots, t_k)) = f^{\mathcal{I}}(\hat{\alpha}(t_1), \ldots, \hat{\alpha}(t_k))$

Note: for constants $\hat{\alpha}(c) = c^{\mathcal{I}}$.

Truth in an interpretation wrt an assignment

We define when a FOL formula φ is true in an interpretation $\mathcal{I}$ wrt an assignment α, written $\mathcal{I}, \alpha \models \varphi$:

- $\mathcal{I}, \alpha \models P(t_1, \ldots, t_k)$ if $(\hat{\alpha}(t_1), \ldots, \hat{\alpha}(t_k)) \in P^{\mathcal{I}}$
- $\mathcal{I}, \alpha \models t_1 = t_2$ if $\hat{\alpha}(t_1) = \hat{\alpha}(t_2)$
- $\mathcal{I}, \alpha \models \neg\varphi$ if $\mathcal{I}, \alpha \not\models \varphi$
- $\mathcal{I}, \alpha \models \varphi \wedge \psi$ if $\mathcal{I}, \alpha \models \varphi$ and $\mathcal{I}, \alpha \models \psi$
- $\mathcal{I}, \alpha \models \varphi \vee \psi$ if $\mathcal{I}, \alpha \models \varphi$ or $\mathcal{I}, \alpha \models \psi$
- $\mathcal{I}, \alpha \models \varphi \rightarrow \psi$ if $\mathcal{I}, \alpha \models \varphi$ implies $\mathcal{I}, \alpha \models \psi$
- $\mathcal{I}, \alpha \models \exists x.\varphi$ if for some $a \in \Delta^{\mathcal{I}}$ we have $\mathcal{I}, \alpha[x \mapsto a] \models \varphi$
- $\mathcal{I}, \alpha \models \forall x.\varphi$ if for every $a \in \Delta^{\mathcal{I}}$ we have $\mathcal{I}, \alpha[x \mapsto a] \models \varphi$

Here, $\alpha[x \mapsto a]$ stands for the new assignment obtained from α as follows:

$$\alpha[x \mapsto a](x) = a$$
$$\alpha[x \mapsto a](y) = \alpha(y) \quad \text{for } y \neq x$$

Open vs. closed formulas

Definitions

- A variable x in a formula φ is free if x does not occur in the scope of any quantifier, otherwise it is bounded.
- An open formula is a formula that has some free variable.
- A closed formula, also called sentence, is a formula that has no free variables.

For closed formulas (but not for open formulas) we can define what it means to be true in an interpretation, written $\mathcal{I} \models \varphi$, without mentioning the assignment, since the assignment α does not play any role in verifying $\mathcal{I}, \alpha \models \varphi$.

Instead, open formulas are strongly related to queries — cf. relational databases.

FOL queries

Def.: A FOL query is an (open) FOL formula.

When φ is a FOL query with free variables $(x_1, \ldots, x_k)$, then we sometimes write it as $\varphi(x_1, \ldots, x_k)$, and say that φ has arity k.

Given an interpretation $\mathcal{I}$, we are interested in those assignments that map the variables $x_1, \ldots, x_k$ (and only those). We write an assignment α s.t. $\alpha(x_i) = a_i$, for $i = 1, \ldots, k$, as $\langle a_1, \ldots, a_k \rangle$.

Def.: Given an interpretation $\mathcal{I}$, the answer to a query $\varphi(x_1, \ldots, x_k)$ is

$$\varphi(x_1, \ldots, x_k)^{\mathcal{I}} = \{(a_1, \ldots, a_k) \mid \mathcal{I}, \langle a_1, \ldots, a_k \rangle \models \varphi(x_1, \ldots, x_k)\}$$

Note: We will also use the notation $\varphi^{\mathcal{I}}$, which keeps the free variables implicit, and $\varphi(\mathcal{I})$ making apparent that φ becomes a functions from interpretations to set of tuples.

FOL boolean queries

Def.: A FOL boolean query is a FOL query without free variables.

Hence, the answer to a boolean query $\varphi()$ is defined as follows:

$$\varphi()^{\mathcal{I}} = \{() \mid \mathcal{I}, \langle\rangle \models \varphi()\}$$

Such an answer is

- $()$, if $\mathcal{I} \models \varphi$
- $\emptyset$, if $\mathcal{I} \not\models \varphi$.

As an obvious convention we read $()$ as "true" and $\emptyset$ as "false".

FOL formulas: logical tasks

Definitions

- Validity: φ is valid iff for all $\mathcal{I}$ and α we have that $\mathcal{I}, \alpha \models \varphi$.
- Satisfiability: φ is satisfiable iff there exists an $\mathcal{I}$ and α such that $\mathcal{I}, \alpha \models \varphi$, and unsatisfiable otherwise.
- Logical implication: φ logically implies ψ, written $\varphi \models \psi$ iff for all $\mathcal{I}$ and α, if $\mathcal{I}, \alpha \models \varphi$ then $\mathcal{I}, \alpha \models \psi$.
- Logical equivalence: φ is logically equivalent to ψ, iff for all $\mathcal{I}$ and α, we have that $\mathcal{I}, \alpha \models \varphi$ iff $\mathcal{I}, \alpha \models \psi$ (i.e., $\varphi \models \psi$ and $\psi \models \varphi$).

FOL queries – Logical tasks

- Validity: if φ is valid, then $\varphi^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \cdots \times \Delta^{\mathcal{I}}$ for all $\mathcal{I}$, i.e., the query always returns all the tuples of $\mathcal{I}$.
- Satisfiability: if φ is satisfiable, then $\varphi^{\mathcal{I}} \neq \emptyset$ for some $\mathcal{I}$, i.e., the query returns at least one tuple.
- Logical implication: if φ logically implies ψ, then $\varphi^{\mathcal{I}} \subseteq \psi^{\mathcal{I}}$ for all $\mathcal{I}$, written $\varphi \subseteq \psi$, i.e., the answer to φ is contained in that of ψ in every interpretation. This is called query containment.
- Logical equivalence: if φ is logically equivalent to ψ, then $\varphi^{\mathcal{I}} = \psi^{\mathcal{I}}$ for all $\mathcal{I}$, written $\varphi \equiv \psi$, i.e., the answer to the two queries is the same in every interpretation. This is called query equivalence and corresponds to query containment in both directions.

Note: These definitions can be extended to the case where we have axioms, i.e., constraints on the admissible interpretations.

Query evaluation

Let us consider:

- a finite alphabet, i.e., we have a finite number of predicates and functions, and
- a finite interpretation $\mathcal{I}$, i.e., an interpretation (over the finite alphabet) for which $\Delta^{\mathcal{I}}$ is finite.

Then we can consider query evaluation as an algorithmic problem, and study its computational properties.

Note: To study the computational complexity of the problem, we need to define a corresponding decision problem.

Query evaluation problem

Definitions

- Query answering problem: given a finite interpretation $\mathcal{I}$ and a FOL query $\varphi(x_1, \ldots, x_k)$, compute

$$\varphi^{\mathcal{I}} = \{(a_1, \ldots, a_k) \mid \mathcal{I}, \langle a_1, \ldots, a_k \rangle \models \varphi(x_1, \ldots, x_k)\}$$

- Recognition problem (for query answering): given a finite interpretation $\mathcal{I}$, a FOL query $\varphi(x_1, \ldots, x_k)$, and a tuple $(a_1, \ldots, a_k)$, with $a_i \in \Delta^{\mathcal{I}}$, check whether $(a_1, \ldots, a_k) \in \varphi^{\mathcal{I}}$, i.e., whether

$$\mathcal{I}, \langle a_1, \ldots, a_k \rangle \models \varphi(x_1, \ldots, x_k)$$

Note: The recognition problem for query answering is the decision problem corresponding to the query answering problem.

Query evaluation algorithm

We define now an algorithm that computes the function $\texttt{Truth}(\mathcal{I}, \alpha, \varphi)$ in such a way that $\texttt{Truth}(\mathcal{I}, \alpha, \varphi) = \texttt{true}$ iff $\mathcal{I}, \alpha \models \varphi$.

We make use of an auxiliary function $\texttt{TermEval}(\mathcal{I}, \alpha, t)$ that, given an interpretation $\mathcal{I}$ and an assignment α, evaluates a term t returning an object $o \in \Delta^{\mathcal{I}}$:

```
Δ^I TermEval(I,α,t) {
   if (t is x ∈ Vars)
      return α(x);
   if (t is f(t_1,...,t_k))
      return f^I(TermEval(I,α,t_1),...,TermEval(I,α,t_k));
}
```

Then, $\texttt{Truth}(\mathcal{I}, \alpha, \varphi)$ can be defined by structural recursion on φ.

Query evaluation algorithm (cont'd)

```
boolean Truth(I,α,φ) {
  if (φ is t_1 = t_2)
    return TermEval(I,α,t_1) = TermEval(I,α,t_2);
  if (φ is P(t_1,...,t_k))
    return P^I(TermEval(I,α,t_1),...,TermEval(I,α,t_k));
  if (φ is ¬ψ)
    return ¬Truth(I,α,ψ);
  if (φ is ψ ∘ ψ')
    return Truth(I,α,ψ) ∘ Truth(I,α,ψ');
  if (φ is ∃x.ψ) {
    boolean b = false;
    for all (a ∈ Δ^I)
       b = b ∨ Truth(I,α[x ↦ a],ψ);
    return b;
  }
  if (φ is ∀x.ψ) {
    boolean b = true;
    for all (a ∈ Δ^I)
       b = b ∧ Truth(I,α[x ↦ a],ψ);
    return b;
  }
}
```

Query evaluation – Results

Theorem (Termination of $\texttt{Truth}(\mathcal{I}, \alpha, \varphi)$)

The algorithm $\texttt{Truth}$ *terminates.*

Proof. Immediate. □

Theorem (Correctness)

The algorithm $\texttt{Truth}$ *is sound and complete, i.e.,* $\mathcal{I}, \alpha \models \varphi$ *if and only if* $\texttt{Truth}(\mathcal{I}, \alpha, \varphi) = \texttt{true}$.

Proof. Easy, since the algorithm is very close to the semantic definition of $\mathcal{I}, \alpha \models \varphi$. □

Query evaluation – Time complexity I

Theorem (Time complexity of $\texttt{Truth}(\mathcal{I}, \alpha, \varphi)$)

The time complexity of $\texttt{Truth}(\mathcal{I}, \alpha, \varphi)$ *is* $O((|\mathcal{I}| + |\alpha| + |\varphi|)^{|\varphi|})$*, i.e., polynomial in the size of* $\mathcal{I}$ *and exponential in the size of* φ.

Proof.

- $f^{\mathcal{I}}$ (of arity k) can be represented as k-dimensional array, hence accessing the required element can be done in time linear in $|\mathcal{I}|$.
- $\texttt{TermEval}(\ldots)$ visits the term, so it generates a linear number of recursive calls, hence its time cost is $O(|\varphi| \cdot (|\mathcal{I}| + |\alpha|))$, i.e., polynomial time in $(|\mathcal{I}| + |\alpha| + |\varphi|)$.
- $P^{\mathcal{I}}$ (of arity k) can be represented as k-dimensional boolean array, hence accessing the required element can be done in time linear in $|\mathcal{I}|$.
- $\texttt{Truth}(\ldots)$ for the boolean cases simply visits the formula, so generates either one or two recursive calls.

Query evaluation – Time complexity II

- `Truth`$(\ldots)$ for the quantified cases $\exists x.\varphi$ and $\forall x.\psi$ involves looping for all elements in $\Delta^{\mathcal{I}}$ and testing the resulting assignments.
- The total number of such testings is $O(|\Delta^{\mathcal{I}}|^{\sharp Vars})$.

Considering that
$O((|\varphi| \cdot (|\mathcal{I}| + |\alpha|)) \cdot |\Delta^{\mathcal{I}}|^{\sharp Vars}) \leq O(|\mathcal{I}| + |\alpha| + |\varphi|)^{(2+|\varphi|)})$, the claim holds. □

Query evaluation – Space complexity I

Theorem (Space complexity of `Truth`$(\mathcal{I}, \alpha, \varphi)$)

The space complexity of `Truth`$(\mathcal{I}, \alpha, \varphi)$ *is* $O(|\varphi| \cdot (|\varphi| \cdot \log |\mathcal{I}|))$*, i.e., logarithmic in the size of* $\mathcal{I}$ *and polynomial in the size of* φ*.*

Proof.

- $f^{\mathcal{I}}(\ldots)$ can be represented as k-dimensional array, hence accessing the required element requires $O(\log |\mathcal{I}|)$;
- `TermEval`$(\ldots)$ simply visits the term, so it generates a linear number of recursive calls. Each activation record has a size $O(\log |\mathcal{I}|)$ to evaluate the function call it represent, and we need $O(|\varphi|)$ activation records;
- $P^{\mathcal{I}}(\ldots)$ can be represented as k-dimensional boolean array, hence accessing the required element requires $O(\log |\mathcal{I}|)$;
- `Truth`$(\ldots)$ for the boolean cases simply visits the formula, so generates either one or two recursive calls, each requiring constant size;
- `Truth`$(\ldots)$ for the quantified cases $\exists x.\varphi$ and $\forall x.\psi$ involves looping for all elements in $\Delta^{\mathcal{I}}$ and testing the resulting assignments;

Query evaluation – Space complexity II

- The total number of activation records that need to be at the same time on the stack is $O(\sharp Vars)$.

Hence, we have $O(\sharp Vars \cdot (|\varphi| \cdot log(|\mathcal{I}|)) \leq O(|\varphi| \cdot (|\varphi| \cdot \log(|\mathcal{I}|))$ the claim holds. □

Note: the worst case form for the formula is

$$\forall x_1.\exists x_2.\ldots\forall x_{n-1}.\exists x_n.P(x_1, x_2, \ldots, x_{n-1}, x_n).$$

Query evaluation – Complexity measures [Var82]

Definition (Combined complexity)

The combined complexity is the complexity of $\{\langle \mathcal{I}, \alpha, \varphi\rangle \mid \mathcal{I}, \alpha \models \varphi\}$, i.e., interpretation, tuple, and query are all considered part of the input.

Definition (Data complexity)

The data complexity is the complexity of $\{\langle \mathcal{I}, \alpha\rangle \mid \mathcal{I}, \alpha \models \varphi\}$, i.e., the query φ is fixed (and hence not considered part of the input).

Definition (Query complexity)

The query complexity is the complexity of $\{\langle \alpha, \varphi\rangle \mid \mathcal{I}, \alpha \models \varphi\}$, i.e., the interpretation $\mathcal{I}$ is fixed (and hence not considered part of the input).

Query evaluation – Combined, data, query complexity

Theorem (Combined complexity of query evaluation)

The complexity of $\{\langle \mathcal{I}, \alpha, \varphi \rangle \mid \mathcal{I}, \alpha \models \varphi\}$ *is:*

- *time: exponential*
- *space:* PSPACE*-complete — see [Var82] for hardness*

Theorem (Data complexity of query evaluation)

The complexity of $\{\langle \mathcal{I}, \alpha \rangle \mid \mathcal{I}, \alpha \models \varphi\}$ *is:*

- *time: polynomial*
- *space:* LOGSPACE

Theorem (Query complexity of query evaluation)

The complexity of $\{\langle \alpha, \varphi \rangle \mid \mathcal{I}, \alpha \models \varphi\}$ *is:*

- *time: exponential*
- *space:* PSPACE*-complete — see [Var82] for hardness*

Example

We consider FOL interpretations exactly as used in relational databases. This requires to drop functions except for constants. Moreover we assume that the interpretation of constants is the identity function, that is constants are interpreted as themselves. This allows us to drop also the interpretation of constants from our interpretations, which now have the form:

$$\mathcal{I} = (\Delta^{\mathcal{I}}, P_1^{\mathcal{I}}, P_2^{\mathcal{I}}, \ldots, P_n^{\mathcal{I}}).$$

Interpretation: $\mathcal{I}$ is as follows (also given in relational notation):

- $\Delta^{\mathcal{I}} = \{john, paul, george, mick, ny, london, 0, 1, \ldots, 100\}$
- $Person^{\mathcal{I}} = \{(john, 30), (paul, 60), (george, 35), (mick, 35)\}$
- $Lives^{\mathcal{I}} = \{(john, ny), (paul, ny), (george, london), (mick, london)\}$
- $Manages^{\mathcal{I}} = \{(paul, john), (george, mick), (paul, mick)\}$

$Person^{\mathcal{I}}$

name	age
john	30
paul	60
george	35
mick	35

$Lives^{\mathcal{I}}$

name	city
john	ny
paul	ny
george	london
mick	london

$Manages^{\mathcal{I}}$

boss	emp. name
paul	john
george	mick
paul	mick

Query: find name and age of persons who live in the same city as their boss.

$$\exists z, w.\mathsf{Person}(x, y) \wedge \mathsf{Manages}(z, x) \wedge \mathsf{Lives}(x, w) \wedge \mathsf{Lives}(z, w)$$

Example - Interpretation

Consider the following interpretation $\mathcal{I}$:

- $\Delta^{\mathcal{I}}$ is equal to the *active domain*: all objects occurring in any predicate extension.
- $Sailors^{\mathcal{I}}$ see table below
- $Boats^{\mathcal{I}}$ see table below
- $Reserves^{\mathcal{I}}$ see table below

$Sailors^{\mathcal{I}}$

sid	sname
22	dustin
31	lubber
58	rusty

$Boats^{\mathcal{I}}$

bid	color
101	red
102	green
103	red
104	blue

$Reserves^{\mathcal{I}}$

sid	bid	day
22	101	10/10/96
58	103	11/12/96

Example - Queries

- Find the names of the sailors who have reserved boat 103.
- Find the names of the sailors who have reserved a red boat.
- Find the colors of the boats reserved by Bob.
- Find the names of the sailors who have reserved at least one boat.
- Find the names of the sailors who have reserved a red and a green boat.
- Find the names of the sailors who have reserved a red or a green boat.
- Find the names of the sailors who have reserved at least two boats.
- Find the names of the sailors who have not reserved a red boat.
- Find the names of the sailors who have reserved all boats.
- Find the names of the sailors who have reserved all red boats.

Example - Queries

- Find the names of the sailors who have reserved boat 103.

Example - Queries

- Find the names of the sailors who have reserved boat 103.

$$\exists x.Sailors(x, y) \wedge \exists w.Reserves(x, 103, z)$$

$$q(y) \leftarrow Sailors(x, y), Reserves(x, 103, z)$$

Example - Queries

- Find the names of the sailors who have reserved boat 103.

$$\exists x.Sailors(x, y) \wedge \exists w.Reserves(x, 103, z)$$

$$q(y) \leftarrow Sailors(x, y), Reserves(x, 103, z)$$

- Find the names of the sailors who have reserved a red boat.

Example - Queries

- Find the names of the sailors who have reserved boat 103.

$$\exists x.Sailors(x, y) \wedge \exists w.Reserves(x, 103, z)$$

$$q(y) \leftarrow Sailors(x, y), Reserves(x, 103, z)$$

- Find the names of the sailors who have reserved a red boat.

$$\exists x.Sailors(x, y) \wedge \exists z, w.Reserves(x, z, w) \wedge Boats(z, red)$$

$$q(y) \leftarrow Sailors(x, y), Reserves(x, z, w), Boats(z, red)$$

Example - Queries

- Find the names of the sailors who have reserved boat 103.

$$\exists x.Sailors(x, y) \wedge \exists w.Reserves(x, 103, z)$$

$$q(y) \leftarrow Sailors(x, y), Reserves(x, 103, z)$$

- Find the names of the sailors who have reserved a red boat.

$$\exists x.Sailors(x, y) \wedge \exists z, w.Reserves(x, z, w) \wedge Boats(z, red)$$

$$q(y) \leftarrow Sailors(x, y), Reserves(x, z, w), Boats(z, red)$$

- Find the colors of the boats reserved by Bob.

Example - Queries

- Find the names of the sailors who have reserved boat 103.

$$\exists x.Sailors(x, y) \wedge \exists w.Reserves(x, 103, z)$$

$$q(y) \leftarrow Sailors(x, y), Reserves(x, 103, z)$$

- Find the names of the sailors who have reserved a red boat.

$$\exists x.Sailors(x, y) \wedge \exists z, w.Reserves(x, z, w) \wedge Boats(z, red)$$

$$q(y) \leftarrow Sailors(x, y), Reserves(x, z, w), Boats(z, red)$$

- Find the colors of the boats reserved by Bob.

$$\exists x.Boats(x, y) \wedge \exists z, w.Reserves(z, y, w) \wedge Sailor(z, Bob)$$

$$q(y) \leftarrow Boats(x, y), Reserves(z, y, w), Sailor(z, Bob)$$

Example - Queries

- Find the names of the sailors who have reserved boat 103.

$$\exists x.Sailors(x, y) \wedge \exists w.Reserves(x, 103, z)$$

$$q(y) \leftarrow Sailors(x, y), Reserves(x, 103, z)$$

- Find the names of the sailors who have reserved a red boat.

$$\exists x.Sailors(x, y) \wedge \exists z, w.Reserves(x, z, w) \wedge Boats(z, red)$$

$$q(y) \leftarrow Sailors(x, y), Reserves(x, z, w), Boats(z, red)$$

- Find the colors of the boats reserved by Bob.

$$\exists x.Boats(x, y) \wedge \exists z, w.Reserves(z, y, w) \wedge Sailor(z, Bob)$$

$$q(y) \leftarrow Boats(x, y), Reserves(z, y, w), Sailor(z, Bob)$$

- Find the names of the sailors who have reserved at least one boat.

Example - Queries

- Find the names of the sailors who have reserved boat 103.

$$\exists x.Sailors(x, y) \wedge \exists w.Reserves(x, 103, z)$$

$$q(y) \leftarrow Sailors(x, y), Reserves(x, 103, z)$$

- Find the names of the sailors who have reserved a red boat.

$$\exists x.Sailors(x, y) \wedge \exists z, w.Reserves(x, z, w) \wedge Boats(z, red)$$

$$q(y) \leftarrow Sailors(x, y), Reserves(x, z, w), Boats(z, red)$$

- Find the colors of the boats reserved by Bob.

$$\exists x.Boats(x, y) \wedge \exists z, w.Reserves(z, y, w) \wedge Sailor(z, Bob)$$

$$q(y) \leftarrow Boats(x, y), Reserves(z, y, w), Sailor(z, Bob)$$

- Find the names of the sailors who have reserved at least one boat.

$$\exists x.Sailors(x, y) \wedge \exists z, w.Reserves(x, z, w)$$

$$q(y) \leftarrow Sailors(x, y), Reserves(x, z, w)$$

Example - Queries

- Find the names of the sailors who have reserved boat 103.

$$\exists x.Sailors(x, y) \wedge \exists w.Reserves(x, 103, z)$$

$$q(y) \leftarrow Sailors(x, y), Reserves(x, 103, z)$$

- Find the names of the sailors who have reserved a red boat.

$$\exists x.Sailors(x, y) \wedge \exists z, w.Reserves(x, z, w) \wedge Boats(z, red)$$

$$q(y) \leftarrow Sailors(x, y), Reserves(x, z, w), Boats(z, red)$$

- Find the colors of the boats reserved by Bob.

$$\exists x.Boats(x, y) \wedge \exists z, w.Reserves(z, y, w) \wedge Sailor(z, Bob)$$

$$q(y) \leftarrow Boats(x, y), Reserves(z, y, w), Sailor(z, Bob)$$

- Find the names of the sailors who have reserved at least one boat.

$$\exists x.Sailors(x, y) \wedge \exists z, w.Reserves(x, z, w)$$

$$q(y) \leftarrow Sailors(x, y), Reserves(x, z, w)$$

- Find the names of the sailors who have reserved a red and a green boat.

Example - Queries

- Find the names of the sailors who have reserved boat 103.

$$\exists x.Sailors(x, y) \wedge \exists w.Reserves(x, 103, z)$$

$$q(y) \leftarrow Sailors(x, y), Reserves(x, 103, z)$$

- Find the names of the sailors who have reserved a red boat.

$$\exists x.Sailors(x, y) \wedge \exists z, w.Reserves(x, z, w) \wedge Boats(z, red)$$

$$q(y) \leftarrow Sailors(x, y), Reserves(x, z, w), Boats(z, red)$$

- Find the colors of the boats reserved by Bob.

$$\exists x.Boats(x, y) \wedge \exists z, w.Reserves(z, y, w) \wedge Sailor(z, Bob)$$

$$q(y) \leftarrow Boats(x, y), Reserves(z, y, w), Sailor(z, Bob)$$

- Find the names of the sailors who have reserved at least one boat.

$$\exists x.Sailors(x, y) \wedge \exists z, w.Reserves(x, z, w)$$

$$q(y) \leftarrow Sailors(x, y), Reserves(x, z, w)$$

- Find the names of the sailors who have reserved a red and a green boat.

$$\exists x.S(x, y) \wedge \exists z, w.R(x, z, w) \wedge B(z, red) \wedge \exists z', w'.R(x, z', w') \wedge B(z', green)$$

Example - Queries

- Find the names of the sailors who have reserved a red or a green boat.

Example - Queries

- Find the names of the sailors who have reserved a red or a green boat.

$$\exists x.Sailors(x, y) \wedge \exists z, w.Reserves(x, z, w) \wedge (Boats(z, red) \vee Boats(z, green))$$

$$q(y) \leftarrow Sailors(x, y), Reserves(x, z, w), Boats(z, red)$$
$$q(y) \leftarrow Sailors(x, y), Reserves(x, z, w), Boats(z, green)$$

Example - Queries

- Find the names of the sailors who have reserved a red or a green boat.

  $\exists x.Sailors(x, y) \wedge \exists z, w.Reserves(x, z, w) \wedge (Boats(z, red) \vee Boats(z, green))$

  $$q(y) \leftarrow Sailors(x, y), Reserves(x, z, w), Boats(z, red)$$
  $$q(y) \leftarrow Sailors(x, y), Reserves(x, z, w), Boats(z, green)$$

- Find the names of the sailors who have reserved at least two boats.

Example - Queries

- Find the names of the sailors who have reserved a red or a green boat.

  $\exists x.Sailors(x, y) \wedge \exists z, w.Reserves(x, z, w) \wedge (Boats(z, red) \vee Boats(z, green))$

  $$q(y) \leftarrow Sailors(x, y), Reserves(x, z, w), Boats(z, red)$$
  $$q(y) \leftarrow Sailors(x, y), Reserves(x, z, w), Boats(z, green)$$

- Find the names of the sailors who have reserved at least two boats.

  $\exists x.Sailors(x, y) \wedge \exists z, w.Reserves(x, z, w) \wedge \exists z', w'.Reserves(x, z', w') \wedge z \neq z'$

Example - Queries

- Find the names of the sailors who have reserved a red or a green boat.

$$\exists x.Sailors(x,y) \wedge \exists z,w.Reserves(x,z,w) \wedge (Boats(z,red) \vee Boats(z,green))$$

$$q(y) \leftarrow Sailors(x,y), Reserves(x,z,w), Boats(z,red)$$
$$q(y) \leftarrow Sailors(x,y), Reserves(x,z,w), Boats(z,green)$$

- Find the names of the sailors who have reserved at least two boats.

$$\exists x.Sailors(x,y) \wedge \exists z,w.Reserves(x,z,w) \wedge \exists z',w'.Reserves(x,z',w') \wedge z \neq z'$$

- Find the names of the sailors who have not reserved a red boat.

Example - Queries

- Find the names of the sailors who have reserved a red or a green boat.

$$\exists x.Sailors(x,y) \wedge \exists z,w.Reserves(x,z,w) \wedge (Boats(z,red) \vee Boats(z,green))$$

$$q(y) \leftarrow Sailors(x,y), Reserves(x,z,w), Boats(z,red)$$
$$q(y) \leftarrow Sailors(x,y), Reserves(x,z,w), Boats(z,green)$$

- Find the names of the sailors who have reserved at least two boats.

$$\exists x.Sailors(x,y) \wedge \exists z,w.Reserves(x,z,w) \wedge \exists z',w'.Reserves(x,z',w') \wedge z \neq z'$$

- Find the names of the sailors who have not reserved a red boat.

$$\exists x.Sailors(x,y) \wedge \forall z,w.(Reserves(x,z,w) \rightarrow \neg Boats(z,red))$$

Example - Queries

- Find the names of the sailors who have reserved a red or a green boat.

$$\exists x.Sailors(x,y) \wedge \exists z,w.Reserves(x,z,w) \wedge (Boats(z,red) \vee Boats(z,green))$$

$$q(y) \leftarrow Sailors(x,y), Reserves(x,z,w), Boats(z,red)$$
$$q(y) \leftarrow Sailors(x,y), Reserves(x,z,w), Boats(z,green)$$

- Find the names of the sailors who have reserved at least two boats.

$$\exists x.Sailors(x,y) \wedge \exists z,w.Reserves(x,z,w) \wedge \exists z',w'.Reserves(x,z',w') \wedge z \neq z'$$

- Find the names of the sailors who have not reserved a red boat.

$$\exists x.Sailors(x,y) \wedge \forall z,w.(Reserves(x,z,w) \rightarrow \neg Boats(z,red))$$

- Find the names of the sailors who have reserved all boats.

Example - Queries

- Find the names of the sailors who have reserved a red or a green boat.

$$\exists x.Sailors(x,y) \wedge \exists z,w.Reserves(x,z,w) \wedge (Boats(z,red) \vee Boats(z,green))$$

$$q(y) \leftarrow Sailors(x,y), Reserves(x,z,w), Boats(z,red)$$
$$q(y) \leftarrow Sailors(x,y), Reserves(x,z,w), Boats(z,green)$$

- Find the names of the sailors who have reserved at least two boats.

$$\exists x.Sailors(x,y) \wedge \exists z,w.Reserves(x,z,w) \wedge \exists z',w'.Reserves(x,z',w') \wedge z \neq z'$$

- Find the names of the sailors who have not reserved a red boat.

$$\exists x.Sailors(x,y) \wedge \forall z,w.(Reserves(x,z,w) \rightarrow \neg Boats(z,red))$$

- Find the names of the sailors who have reserved all boats.

$$\exists x.Sailors(x,y) \wedge \forall z,c.(Boats(s,c) \rightarrow \exists w.Reserves(x,z,w))$$

Example - Queries

- Find the names of the sailors who have reserved a red or a green boat.

$$\exists x.Sailors(x,y) \wedge \exists z,w.Reserves(x,z,w) \wedge (Boats(z,red) \vee Boats(z,green))$$

$$q(y) \leftarrow Sailors(x,y), Reserves(x,z,w), Boats(z,red)$$
$$q(y) \leftarrow Sailors(x,y), Reserves(x,z,w), Boats(z,green)$$

- Find the names of the sailors who have reserved at least two boats.

$$\exists x.Sailors(x,y) \wedge \exists z,w.Reserves(x,z,w) \wedge \exists z',w'.Reserves(x,z',w') \wedge z \neq z'$$

- Find the names of the sailors who have not reserved a red boat.

$$\exists x.Sailors(x,y) \wedge \forall z,w.(Reserves(x,z,w) \rightarrow \neg Boats(z,red))$$

- Find the names of the sailors who have reserved all boats.

$$\exists x.Sailors(x,y) \wedge \forall z,c.(Boats(s,c) \rightarrow \exists w.Reserves(x,z,w))$$

- Find the names of the sailors who have reserved all red boats.

Example - Queries

- Find the names of the sailors who have reserved a red or a green boat.

$$\exists x.Sailors(x,y) \wedge \exists z,w.Reserves(x,z,w) \wedge (Boats(z,red) \vee Boats(z,green))$$

$$q(y) \leftarrow Sailors(x,y), Reserves(x,z,w), Boats(z,red)$$
$$q(y) \leftarrow Sailors(x,y), Reserves(x,z,w), Boats(z,green)$$

- Find the names of the sailors who have reserved at least two boats.

$$\exists x.Sailors(x,y) \wedge \exists z,w.Reserves(x,z,w) \wedge \exists z',w'.Reserves(x,z',w') \wedge z \neq z'$$

- Find the names of the sailors who have not reserved a red boat.

$$\exists x.Sailors(x,y) \wedge \forall z,w.(Reserves(x,z,w) \rightarrow \neg Boats(z,red))$$

- Find the names of the sailors who have reserved all boats.

$$\exists x.Sailors(x,y) \wedge \forall z,c.(Boats(s,c) \rightarrow \exists w.Reserves(x,z,w))$$

- Find the names of the sailors who have reserved all red boats.

$$\exists x.Sailors(x,y) \wedge \forall z.(Boats(z,red) \rightarrow \exists w.Reserves(x,z,w))$$

Conjunctive queries (CQs)

Def.: A conjunctive query (CQ) is a FOL query of the form

$$\exists \vec{y}.conj(\vec{x}, \vec{y})$$

where $conj(\vec{x}, \vec{y})$ is a conjunction (i.e., an "and") of atoms and equalities, over the free variables $\vec{x}$, the existentially quantified variables $\vec{y}$, and possibly constants.

Note:

- CQs contain no disjunction, no negation, no universal quantification, and no function symbols besides constants.
- Hence, they correspond to relational algebra select-project-join (SPJ) queries.
- CQs are the most frequently asked queries.

Conjunctive queries and SQL – Example

Relational alphabet:
Person(name, age), Lives(person, city), Manages(boss, employee)

Query: find the name and the age of the persons who live in the same city as their boss.

Conjunctive queries and SQL – Example

Relational alphabet:
Person(name, age), Lives(person, city), Manages(boss, employee)

Query: find the name and the age of the persons who live in the same city as their boss.

Expressed in SQL:

```
SELECT P.name, P.age
FROM Person P,  Manages M,  Lives L1,  Lives L2
WHERE P.name = L1.person  AND  P.name = M.employee  AND
      M.boss = L2.person  AND  L1.city = L2.city
```

Conjunctive queries and SQL – Example

Relational alphabet:
Person(name, age), Lives(person, city), Manages(boss, employee)

Query: find the name and the age of the persons who live in the same city as their boss.

Expressed in SQL:

```
SELECT P.name, P.age
FROM Person P,  Manages M,  Lives L1,  Lives L2
WHERE P.name = L1.person  AND  P.name = M.employee  AND
      M.boss = L2.person  AND  L1.city = L2.city
```

Expressed as a CQ:

$$\exists b, e, p_1, c_1, p_2, c_2.\text{Person}(n, a) \wedge \text{Manages}(b, e) \wedge \text{Lives}(p1, c1) \wedge \text{Lives}(p2, c2) \wedge n = p1 \ \wedge \ n = e \ \wedge \ b = p2 \ \wedge \ c1 = c2$$

Conjunctive queries and SQL – Example

Relational alphabet:
$\text{Person}(\text{name}, \text{age})$, $\text{Lives}(\text{person}, \text{city})$, $\text{Manages}(\text{boss}, \text{employee})$

Query: find the name and the age of the persons who live in the same city as their boss.

Expressed in SQL:

```
SELECT P.name, P.age
FROM Person P,  Manages M,  Lives L1,  Lives L2
WHERE P.name = L1.person  AND  P.name = M.employee  AND
      M.boss = L2.person  AND  L1.city = L2.city
```

Expressed as a CQ:

$\exists b, e, p_1, c_1, p_2, c_2.\text{Person}(n, a) \wedge \text{Manages}(b, e) \wedge \text{Lives}(p1, c1) \wedge \text{Lives}(p2, c2) \wedge$
$n = p1 \ \wedge \ n = e \ \wedge \ b = p2 \ \wedge \ c1 = c2$

Or simpler: $\exists b, c.\text{Person}(n, a) \wedge \text{Manages}(b, n) \wedge \text{Lives}(n, c) \wedge \text{Lives}(b, c)$

Datalog notation for CQs

A CQ $q = \exists \vec{y}.conj(\vec{x}, \vec{y})$ can also be written using datalog notation as

$$q(\vec{x}_1) \leftarrow conj'(\vec{x}_1, \vec{y}_1)$$

where $conj'(\vec{x}_1, \vec{y}_1)$ is the list of atoms in $conj(\vec{x}, \vec{y})$ obtained by equating the variables $\vec{x}$, $\vec{y}$ according to the equalities in $conj(\vec{x}, \vec{y})$.

As a result of such an equality elimination, we have that $\vec{x}_1$ and $\vec{y}_1$ can contain constants and multiple occurrences of the same variable.

Def.: In the above query q, we call:

- $q(\vec{x}_1)$ the head;
- $conj'(\vec{x}_1, \vec{y}_1)$ the body;
- the variables in $\vec{x}_1$ the distinguished variables;
- the variables in $\vec{y}_1$ the non-distinguished variables.

Conjunctive queries – Example

- Consider an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, E^{\mathcal{I}})$, where $E^{\mathcal{I}}$ is a binary relation – *note that such interpretation is a (directed) graph.*
- The following CQ q returns all nodes that participate to a triangle in the graph:

$$\exists y, z.E(x, y) \wedge E(y, z) \wedge E(z, x)$$

- The query q in datalog notation becomes:

$$q(x) \leftarrow E(x, y), E(y, z), E(z, x)$$

- The query q in SQL is (we use `Edge(f,s)` for $E(x, y)$):

```
SELECT E1.f
FROM Edge E1, Edge E2, Edge E3
WHERE E1.s = E2.f AND E2.s = E3.f AND E3.s = E1.f
```

Nondeterministic evaluation of CQs

Since a CQ contains only existential quantifications, we can evaluate it by:

1. guessing a truth assignment for the non-distinguished variables;
2. evaluating the resulting formula (that has no quantifications).

```
boolean ConjTruth(I, α, ∃y⃗.conj(x⃗, y⃗)) {
  GUESS assignment α[y⃗ ↦ a⃗] {
     return Truth(I, α[y⃗ ↦ a⃗], conj(x⃗, y⃗));
}
```

where $\texttt{Truth}(\mathcal{I}, \alpha, \varphi)$ is defined as for FOL queries, considering only the required cases.

Nondeterministic CQ evaluation algorithm

```
boolean Truth(I,α,φ) {
  if (φ is t_1 = t_2)
    return TermEval(I,α,t_1) = TermEval(I,α,t_2);
  if (φ is P(t_1,...,t_k))
    return P^I(TermEval(I,α,t_1),...,TermEval(I,α,t_k));
  if (φ is ψ ∧ ψ')
    return Truth(I,α,ψ) ∧ Truth(I,α,ψ');
}

Δ^I TermEval(I,α,t) {
   if (t is a variable x) return α(x);
   if (t is a constant c) return c^I;
}
```

CQ evaluation – Combined, data, and query complexity

Theorem (Combined complexity of CQ evaluation)

$\{\langle \mathcal{I}, \alpha, q\rangle \mid \mathcal{I}, \alpha \models q\}$ *is NP-complete — see below for hardness*

- *time: exponential*
- *space: polynomial*

Theorem (Data complexity of CQ evaluation)

$\{\langle \mathcal{I}, \alpha\rangle \mid \mathcal{I}, \alpha \models q\}$ *is* LogSpace

- *time: polynomial*
- *space: logarithmic*

Theorem (Query complexity of CQ evaluation)

$\{\langle \alpha, q\rangle \mid \mathcal{I}, \alpha \models q\}$ *is NP-complete — see below for hardness*

- *time: exponential*
- *space: polynomial*

3-colorability

A graph is *k*-colorable if it is possible to assign to each node one of *k* colors in such a way that every two nodes connected by an edge have different colors.

Def.: 3-colorability is the following decision problem

Given a graph $G = (V, E)$, is it 3-colorable?

Theorem

3-colorability is NP-complete.

3-colorability

A graph is *k*-colorable if it is possible to assign to each node one of *k* colors in such a way that every two nodes connected by an edge have different colors.

Def.: 3-colorability is the following decision problem

Given a graph $G = (V, E)$, is it 3-colorable?

Theorem

3-colorability is NP-complete.

We exploit 3-colorability to show NP-hardness of conjunctive query evaluation.

Reduction from 3-colorability to CQ evaluation

Let $G = (V, E)$ be a graph. We define:

- An Interpretation: $\mathcal{I} = (\Delta^{\mathcal{I}}, E^{\mathcal{I}})$ where:
  - $\Delta^{\mathcal{I}} = \{\mathsf{r}, \mathsf{g}, \mathsf{b}\}$
  - $E^{\mathcal{I}} = \{(\mathsf{r}, \mathsf{g}), (\mathsf{g}, \mathsf{r}), (\mathsf{r}, \mathsf{b}), (\mathsf{b}, \mathsf{r}), (\mathsf{g}, \mathsf{b}), (\mathsf{b}, \mathsf{g})\}$
- A conjunctive query: Let $V = \{x_1, \ldots, x_n\}$, then consider the boolean conjunctive query defined as:

$$q_G \;=\; \exists x_1, \ldots, x_n. \bigwedge_{(x_i, x_j) \in E} E(x_i, x_j) \wedge E(x_j, x_i)$$

Theorem
G is 3-colorable iff $\mathcal{I} \models q_G$.

NP-hardness of CQ evaluation

The previous reduction immediately gives us the hardness for combined complexity.

Theorem
CQ evaluation is NP-hard in combined complexity.

NP-hardness of CQ evaluation

The previous reduction immediately gives us the hardness for combined complexity.

Theorem
CQ evaluation is NP-hard in combined complexity.

Note: in the previous reduction, the interpretation does not depend on the actual graph. Hence, the reduction provides also the lower-bound for query complexity.

Theorem
CQ evaluation is NP-hard in query (and combined) complexity.

Recognition problem and boolean query evaluation

Consider the recognition problem associated to the evaluation of a query q of arity k. Then

$$\mathcal{I}, \alpha \models q(x_1, \ldots, x_k) \qquad \text{iff} \qquad \mathcal{I}_{\alpha,\vec{c}} \models q(c_1, \ldots, c_k)$$

where $\mathcal{I}_{\alpha,\vec{c}}$ is identical to $\mathcal{I}$ but includes new constants $c_1, \ldots, c_k$ that are interpreted as $c_i^{\mathcal{I}_{\alpha,\vec{c}}} = \alpha(x_i)$.

That is, we can reduce the recognition problem to the evaluation of a boolean query.

Homomorphism

Let $\mathcal{I} = (\Delta^{\mathcal{I}}, P^{\mathcal{I}}, \ldots, c^{\mathcal{I}}, \ldots)$ and $\mathcal{J} = (\Delta^{\mathcal{J}}, P^{\mathcal{J}}, \ldots, c^{\mathcal{J}}, \ldots)$ be two interpretations over the same alphabet (for simplicity, we consider only constants as functions).

Def.: A homomorphism from $\mathcal{I}$ to $\mathcal{J}$

is a mapping $h : \Delta^{\mathcal{I}} \to \Delta^{\mathcal{J}}$ such that:

- $h(c^{\mathcal{I}}) = c^{\mathcal{J}}$
- $(o_1, \ldots, o_k) \in P^{\mathcal{I}}$ implies $(h(o_1), \ldots, h(o_k)) \in P^{\mathcal{J}}$

Note: An isomorphism is a homomorphism that is one-to-one and onto.

Theorem

FOL is unable to distinguish between interpretations that are isomorphic.

Proof. See any standard book on logic. □

Canonical interpretation of a (boolean) CQ

Let q be a conjunctive query $\exists x_1, \ldots, x_n.conj$

Def.: The canonical interpretation $\mathcal{I}_q$ associated with q

is the interpretation $\mathcal{I}_q = (\Delta^{\mathcal{I}_q}, P^{\mathcal{I}_q}, \ldots, c^{\mathcal{I}_q}, \ldots)$, where

- $\Delta^{\mathcal{I}_q} = \{x_1, \ldots, x_n\} \cup \{c \mid c \text{ constant occurring in } q\}$, i.e., all the variables and constants in q;
- $c^{\mathcal{I}_q} = c$, for each constant c in q;
- $(t_1, \ldots, t_k) \in P^{\mathcal{I}_q}$ iff the atom $P(t_1, \ldots, t_k)$ occurs in q.

Canonical interpretation of a (boolean) CQ – Example

Consider the boolean query q

$$q(c) \leftarrow E(c, y), E(y, z), E(z, c)$$

Then, the canonical interpretation $\mathcal{I}_q$ is defined as

$$\mathcal{I}_q \;=\; (\Delta^{\mathcal{I}_q}, E^{\mathcal{I}_q}, c^{\mathcal{I}_q})$$

where

- $\Delta^{\mathcal{I}_q} = \{y, z, c\}$
- $E^{\mathcal{I}_q} = \{(c, y), (y, z), (z, c)\}$
- $c^{\mathcal{I}_q} = c$

Homomorphism theorem

Theorem ([CM77])

For boolean CQs, $\mathcal{I} \models q$ iff there exists a homomorphism from $\mathcal{I}_q$ to $\mathcal{I}$.

Proof.
"$\Rightarrow$" Let $\mathcal{I} \models q$, let α be an assignment to the existential variables that makes q true in $\mathcal{I}$, and let $\hat{\alpha}$ be its extension to constants. Then $\hat{\alpha}$ is a homomorphism from $\mathcal{I}_q$ to $\mathcal{I}$.

"$\Leftarrow$" Let h be a homomorphism from $\mathcal{I}_q$ to $\mathcal{I}$. Then restricting h to the variables only we obtain an assignment to the existential variables that makes q true in $\mathcal{I}$. □

Illustration of homomorphism theorem – Interpretation

Consider the following interpretation $\mathcal{I}$:

- $\Delta^{\mathcal{I}} = \{john, paul, george, mick, ny, london, 0, \ldots, 110\}$
- $Person^{\mathcal{I}} = \{(john, 30), (paul, 60), (george, 35), (mick, 35)\}$
- $Lives^{\mathcal{I}} = \{(john, ny), (paul, ny), (george, london), (mick, london)\}$
- $Manages^{\mathcal{I}} = \{(paul, john), (george, mick), (paul, mick)\}$

In relational notation:

$Person^{\mathcal{I}}$

name	age
john	30
paul	60
george	35
mick	35

$Lives^{\mathcal{I}}$

name	city
john	ny
paul	ny
george	london
mick	london

$Manages^{\mathcal{I}}$

boss	emp. name
paul	john
george	mick
paul	mick

Illustration of homomorphism theorem – Query

Consider the following query q:

$$q() \leftarrow Person(john, z), Manages(x, john), Lives(x, y), Lives(john, y)$$

"There exists a manager that has john as an employee and lives in the same city of him?"
The canonical model $\mathcal{I}_q$ is:

- $\Delta^{\mathcal{I}} = \{john, x, y, z\}$
- $john^{\mathcal{I}} = john$
- $Person^{\mathcal{I}_q} = \{(john, z)\}$
- $Lives^{\mathcal{I}_q} = \{(john, y), (x, y)\}$
- $Manages^{\mathcal{I}_q} = \{(x, john)\}$

In relational notation:

$Person^{\mathcal{I}_q}$

name	age
john	z

$Lives^{\mathcal{I}_q}$

name	city
john	y
x	y

$Manages^{\mathcal{I}_q}$

boss	emp. name
x	john

Illustration of homomorphism theorem – If-direction

Hp: $\mathcal{I} \models q$. **Th**: There exists an homomrphism $h : \mathcal{I}_q \to \mathcal{I}$.
If $\mathcal{I} \models q$, then there exists an assignment $\hat{\alpha}$ such that $\langle \mathcal{I}, \alpha \rangle \models \mathit{body}(q)$:

- $\alpha(x) = \mathit{paul}$
- $\alpha(z) = 30$
- $\alpha(y) = \mathit{ny}$

Let us extend $\hat{\alpha}$ to constants:

- $\hat{\alpha}(\mathit{john}) = \mathit{john}$

$h = \hat{\alpha}$ is an homomorphism from $\mathcal{I}_{q_1}$ to $\mathcal{I}$:

- $h(\mathit{john}^{\mathcal{I}_q}) = \mathit{john}^{\mathcal{I}}$? Yes!
- $(\mathit{john}, z)) \in \mathit{Person}^{\mathcal{I}_q}$ implies $(h(\mathit{john}), h(z)) \in \mathit{Person}^{\mathcal{I}}$?
  Yes: $(\mathit{john}, 30) \in \mathit{Person}^{\mathcal{I}}$;
- $(\mathit{john}, x) \in \mathit{Lives}^{\mathcal{I}_q}$ implies $h(\mathit{john}), h(x)) \in \mathit{Lives}^{\mathcal{I}}$?
  Yes: $(\mathit{john}, \mathit{ny}) \in \mathit{Lives}^{\mathcal{I}}$;
- $(x, y) \in \mathit{Lives}^{\mathcal{I}_q}$ implies $(h(x), h(y)) \in \mathit{Lives}^{\mathcal{I}}$?
  Yes: $(\mathit{paul}, \mathit{ny}) \in \mathit{Lives}^{\mathcal{I}}$;
- $(x, \mathit{john}) \in \mathit{Manages}^{\mathcal{I}_q}$ implies $(h(x), h(\mathit{john})) \in \mathit{Manages}^{\mathcal{I}}$?
  Yes: $(\mathit{paul}, \mathit{john}) \in \mathit{Manages}^{\mathcal{I}}$.

Illustration of homomorphism theorem – Only-if-direction

Hp: There exists an homomrphism $h : \mathcal{I}_q \to \mathcal{I}$. **Th**: $\mathcal{I} \models q$.
Let $h : \mathcal{I}_q \to \mathcal{I}$:

- $h(\mathit{john}) = \mathit{john}$;
- $h(x) = \mathit{paul}$;
- $h(z) = 30$;
- $h(y) = \mathit{ny}$.

Let us define an assignment α by restricting h to variables:

- $\alpha(x) = \mathit{paul}$;
- $\alpha(z) = 30$;
- $\alpha(y) = \mathit{ny}$.

Then $\langle \mathcal{I}, \alpha \rangle \models \mathit{body}(q)$. Indeed:

- $(\mathit{john}, \alpha(z)) = (\mathit{john}, 30) \in \mathit{Person}^{\mathcal{I}}$;
- $(\alpha(x), \mathit{john}) = (\mathit{paul}, \mathit{john}) \in \mathit{Manages}^{\mathcal{I}}$;
- $(\alpha(x), \alpha(y)) = (\mathit{paul}, \mathit{ny}) \in \mathit{Lives}^{\mathcal{I}}$;
- $(\mathit{john}, \alpha(y)) = (\mathit{john}, \mathit{ny}) \in \mathit{Lives}^{\mathcal{I}}$.

Canonical interpretation and (boolean) CQ evaluation

The previous result can be rephrased as follows:

(The recognition problem associated to) query evaluation can be reduced to finding a homomorphism.

Finding a homomorphism between two interpretations (aka relational structures) is also known as solving a Constraint Satisfaction Problem (CSP), a problem well-studied in AI – see also [KV98].

Observations

Theorem

$\mathcal{I}_q \models q$ *is always true.*

Proof. By Chandra Merlin theorem: $\mathcal{I}_q \models q$ iff there exists homomorph. from $\mathcal{I}_q$ to $\mathcal{I}_q$. Identity is one such homomorphism. □

Theorem

Let h be a homomorphism from $\mathcal{I}_1$ to $\mathcal{I}_2$, and h' be a homomorphism from $\mathcal{I}_2$ to $\mathcal{I}_3$. Then $h \circ h'$ is a homomorphism form $\mathcal{I}_1$ to $\mathcal{I}_3$.

Proof. Just check that $h \circ h'$ satisfied the definition of homomorphism: i.e. $h'(h(\cdot))$is a mapping from $\Delta^{\mathcal{I}_1}$ to $\Delta^{\mathcal{I}_3}$ such that:

- $h'(h(c^{\mathcal{I}_1})) = c^{\mathcal{I}_3}$;
- $(o_1, \ldots, o_k) \in P^{\mathcal{I}_1}$ implies $(h'(h(o_1)), \ldots, h'(h(o_k))) \in P^{\mathcal{I}_3}$. □

The CQs characterizing property

Def.: Homomorphic equivalent interpretations

Two interpretations $\mathcal{I}$ and $\mathcal{J}$ are homomorphically equivalent if there is homomorphism $h_{\mathcal{I},\mathcal{J}}$ from $\mathcal{I}$ to $\mathcal{J}$ and homomorphism $h_{\mathcal{J},\mathcal{I}}$ from $\mathcal{J}$ to $\mathcal{I}$.

Theorem (model theoretic characterization of CQs)

CQs are unable to distinguish between interpretations that are homomorphic equivalent.

Proof. Consider any two homomorphically equivalent interpretations $\mathcal{I}$ and $\mathcal{J}$ with homomorphism $h_{\mathcal{I},\mathcal{J}}$ from $\mathcal{I}$ to $\mathcal{J}$ and homomorphism $h_{\mathcal{J},\mathcal{I}}$ from $\mathcal{J}$ to $\mathcal{I}$.

- If $\mathcal{I} \models q$ then there exists a homomorphism h from $\mathcal{I}_q$ to $\mathcal{I}$. But then $h \circ h_{\mathcal{I},\mathcal{J}}$ is an hom form $\mathcal{I}_q$ to $\mathcal{J}$, hence $\mathcal{J} \models q$.
- Similarly, if $\mathcal{J} \models q$ then there exists a homomorph. g from $\mathcal{I}_q$ to $\mathcal{J}$. But then $g \circ h_{\mathcal{J},\mathcal{I}}$ is a homomorph. form $\mathcal{I}_q$ to $\mathcal{I}$, hence $\mathcal{I} \models q$. □

Query containment

Def.: Query containment

Given two FOL queries φ and ψ of the same arity, φ is contained in ψ, denoted $\varphi \subseteq \psi$, if for all interpretations $\mathcal{I}$ and all assignments α we have that

$$\mathcal{I}, \alpha \models \varphi \quad \text{implies} \quad \mathcal{I}, \alpha \models \psi$$

(In logical terms: $\varphi \models \psi$.)

Note: Query containment is of special interest in query optimization.

Query containment

Def.: Query containment

Given two FOL queries φ and ψ of the same arity, φ is contained in ψ, denoted $\varphi \subseteq \psi$, if for all interpretations $\mathcal{I}$ and all assignments α we have that

$$\mathcal{I}, \alpha \models \varphi \quad \text{implies} \quad \mathcal{I}, \alpha \models \psi$$

(In logical terms: $\varphi \models \psi$.)

Note: Query containment is of special interest in query optimization.

Theorem

For FOL queries, query containment is undecidable.

Proof.: Reduction from FOL logical implication. □

Query containment for CQs

For CQs, query containment $q_1(\vec{x}) \subseteq q_2(\vec{x})$ can be reduced to query evaluation.

1. Freeze the free variables, i.e., consider them as constants.
   This is possible, since $q_1(\vec{x}) \subseteq q_2(\vec{x})$ iff
   - $\mathcal{I}, \alpha \models q_1(\vec{x})$ implies $\mathcal{I}, \alpha \models q_2(\vec{x})$, for all $\mathcal{I}$ and α; or equivalently
   - $\mathcal{I}_{\alpha,\vec{c}} \models q_1(\vec{c})$ implies $\mathcal{I}_{\alpha,\vec{c}} \models q_2(\vec{c})$, for all $\mathcal{I}_{\alpha,\vec{c}}$, where $\vec{c}$ are new constants, and $\mathcal{I}_{\alpha,\vec{c}}$ extends $\mathcal{I}$ to the new constants with $c^{\mathcal{I}_{\alpha,\vec{c}}} = \alpha(x)$.

2. Construct the canonical interpretation $\mathcal{I}_{q_1(\vec{c})}$ of the CQ $q_1(\vec{c})$ on the left hand side ...

3. ...and evaluate on $\mathcal{I}_{q_1(\vec{c})}$ the CQ $q_2(\vec{c})$ on the right hand side, i.e., check whether $\mathcal{I}_{q_1(\vec{c})} \models q_2(\vec{c})$.

Reducing containment of CQs to CQ evaluation

Theorem ([CM77])

For CQs, $q_1(\vec{x}) \subseteq q_2(\vec{x})$ iff $\mathcal{I}_{q_1(\vec{c})} \models q_2(\vec{c})$, where $\vec{c}$ are new constants.

Proof.

"$\Rightarrow$" Assume that $q_1(\vec{x}) \subseteq q_2(\vec{x})$.

- Since $\mathcal{I}_{q_1(\vec{c})} \models q_1(\vec{c})$ it follows that $\mathcal{I}_{q_1(\vec{c})} \models q_2(\vec{c})$.

"$\Leftarrow$" Assume that $\mathcal{I}_{q_1(\vec{c})} \models q_2(\vec{c})$.

- By [CM77] on hom., for every $\mathcal{I}$ such that $\mathcal{I} \models q_1(\vec{c})$ there exists a homomorphism h from $\mathcal{I}_{q_1(\vec{c})}$ to $\mathcal{I}$.
- On the other hand, since $\mathcal{I}_{q_1(\vec{c})} \models q_2(\vec{c})$, again by [CM77] on hom., there exists a homomorphism h' from $\mathcal{I}_{q_2(\vec{c})}$ to $\mathcal{I}_{q_1(\vec{c})}$.
- The mapping $h \circ h'$ (obtained by composing h and h') is a homomorphism from $\mathcal{I}_{q_2(\vec{c})}$ to $\mathcal{I}$. Hence, once again by [CM77] on hom., $\mathcal{I} \models q_2(\vec{c})$.

So we can conclude that $q_1(\vec{c}) \subseteq q_2(\vec{c})$, and hence $q_1(\vec{x}) \subseteq q_2(\vec{x})$. $\square$

Query containment for CQs

For CQs, we also have that (boolean) query evaluation $\mathcal{I} \models q$ can be reduced to query containment.

Let $\mathcal{I} = (\Delta^{\mathcal{I}}, P^{\mathcal{I}}, \ldots, c^{\mathcal{I}}, \ldots)$.
We construct the (boolean) CQ $q_{\mathcal{I}}$ as follows:

- $q_{\mathcal{I}}$ has no existential variables (hence no variables at all);
- the constants in $q_{\mathcal{I}}$ are the elements of $\Delta^{\mathcal{I}}$;
- for each relation P interpreted in $\mathcal{I}$ and for each fact $(a_1, \ldots, a_k) \in P^{\mathcal{I}}$, $q_{\mathcal{I}}$ contains one atom $P(a_1, \ldots, a_k)$ (note that each $a_i \in \Delta^{\mathcal{I}}$ is a constant in $q_{\mathcal{I}}$).

Theorem

For CQs, $\mathcal{I} \models q$ iff $q_{\mathcal{I}} \subseteq q$.

Query containment for CQs – Complexity

From the previous results and NP-completenss of combined complexity of CQ evaluation, we immediately get:

Theorem

Containment of CQs is NP-complete.

Query containment for CQs – Complexity

From the previous results and NP-completenss of combined complexity of CQ evaluation, we immediately get:

Theorem

Containment of CQs is NP-complete.

Since CQ evaluation is NP-complete even in query complexity, the above result can be strengthened:

Theorem

Containment $q_1(\vec{x}) \subseteq q_2(\vec{x})$ of CQs is NP-complete, even when q_1 is considered fixed.

Union of conjunctive queries (UCQs)

Def.: A union of conjunctive queries (UCQ) is a FOL query of the form

$$\bigvee_{i=1,\ldots,n} \exists \vec{y}_i . conj_i(\vec{x}, \vec{y}_i)$$

where each $conj_i(\vec{x}, \vec{y}_i)$ is a conjunction of atoms and equalities with free variables $\vec{x}$ and $\vec{y}_i$, and possibly constants.

Note: Obviously, each conjunctive query is also a of union of conjunctive queries.

Datalog notation for UCQs

A union of conjunctive queries

$$q = \bigvee_{i=1,\ldots,n} \exists \vec{y}_i . conj_i(\vec{x}, \vec{y}_i)$$

is written in datalog notation as

$$\begin{array}{lcl} \{ \; q(\vec{x}) & \leftarrow & conj'_1(\vec{x}, \vec{y_1}') \\ & \vdots & \\ q(\vec{x}) & \leftarrow & conj'_n(\vec{x}, \vec{y_n}') \; \} \end{array}$$

where each element of the set is the datalog expression corresponding to the conjunctive query $q_i = \exists \vec{y}_i . conj_i(\vec{x}, \vec{y}_i)$.

Note: in general, we omit the set brackets.

Evaluation of UCQs

From the definition of FOL query we have that:

$$\mathcal{I}, \alpha \models \bigvee_{i=1,\ldots,n} \exists \vec{y}_i . conj_i(\vec{x}, \vec{y}_i)$$

if and only if

$$\mathcal{I}, \alpha \models \exists \vec{y}_i . conj_i(\vec{x}, \vec{y}_i) \qquad \text{for some } i \in \{1, \ldots, n\}.$$

Hence to evaluate a UCQ q, we simply evaluate a number (linear in the size of q) of conjunctive queries in isolation.

Hence, evaluating UCQs has the same complexity as evaluating CQs.

UCQ evaluation – Combined, data, and query complexity

Theorem (Combined complexity of UCQ evaluation)

$\{\langle \mathcal{I}, \alpha, q \rangle \mid \mathcal{I}, \alpha \models q\}$ *is NP-complete.*

- *time: exponential*
- *space: polynomial*

Theorem (Data complexity of UCQ evaluation)

$\{\langle \mathcal{I}, q \rangle \mid \mathcal{I}, \alpha \models q\}$ *is* LogSpace*-complete (query q fixed).*

- *time: polynomial*
- *space: logarithmic*

Theorem (Query complexity of UCQ evaluation)

$\{\langle \alpha, q \rangle \mid \mathcal{I}, \alpha \models q\}$ *is NP-complete (interpretation $\mathcal{I}$ fixed).*

- *time: exponential*
- *space: polynomial*

Query containment for UCQs

Theorem
For UCQs, $\{q_1, \ldots, q_k\} \subseteq \{q'_1, \ldots, q'_n\}$ iff for each q_i there is a q'_j such that $q_i \subseteq q'_j$.

Proof.
"$\Leftarrow$" Obvious.

"$\Rightarrow$" If the containment holds, then we have $\{q_1(\vec{c}), \ldots, q_k(\vec{c})\} \subseteq \{q'_1(\vec{c}), \ldots, q'_n(\vec{c})\}$, where $\vec{c}$ are new constants:

- Now consider $\mathcal{I}_{q_i(\vec{c})}$. We have $\mathcal{I}_{q_i(\vec{c})} \models q_i(\vec{c})$, and hence $\mathcal{I}_{q_i(\vec{c})} \models \{q_1(\vec{c}), \ldots, q_k(\vec{c})\}$.
- By the containment, we have that $\mathcal{I}_{q_i(\vec{c})} \models \{q'_1(\vec{c}), \ldots, q'_n(\vec{c})\}$. I.e., there exists a $q'_j(\vec{c})$ such that $\mathcal{I}_{q_i(\vec{c})} \models q'_j(\vec{c})$.
- Hence, by [CM77] on containment of CQs, we have $q_i \subseteq q'_j$. □

Query containment for UCQs – Complexity

From the previous result, we have that we can check $\{q_1, \ldots, q_k\} \subseteq \{q'_1, \ldots, q'_n\}$ by at most $k \cdot n$ CQ containment checks.

We immediately get:

Theorem
Containment of UCQs is NP-complete.

References

[CM77] A. K. Chandra and P. M. Merlin.
Optimal implementation of conjunctive queries in relational data bases.
In *Proc. of the 9th ACM Symp. on Theory of Computing (STOC'77)*, pages 77–90, 1977.

[KV98] P. G. Kolaitis and M. Y. Vardi.
Conjunctive-query containment and constraint satisfaction.
In *Proc. of the 17th ACM SIGACT SIGMOD SIGART Symp. on Principles of Database Systems (PODS'98)*, pages 205–213, 1998.

[Var82] M. Y. Vardi.
The complexity of relational query languages.
In *Proc. of the 14th ACM SIGACT Symp. on Theory of Computing (STOC'82)*, pages 137–146, 1982.