Robotics 2

September 19, 2024

Exercise 1

Consider the robot in Fig. 1, with a first revolute and a second prismatic joint, moving in a vertical plane. The joint coordinates q_1 and q_2 to be used are also shown in the figure. Friction at the joints can be neglected.

a) Derive the dynamic model of the robot in the Lagrangian form $\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$. If needed, introduce any missing kinematic or dynamic parameter.

b) Find a linear parametrization $\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\, \boldsymbol{a} = \boldsymbol{\tau}$ of the robot dynamics in terms of a vector $\boldsymbol{a} \in \mathbb{R}^r$ of dynamic coefficients and a $2 \times r$ regressor matrix $\boldsymbol{Y}$. Discuss the minimality of r.

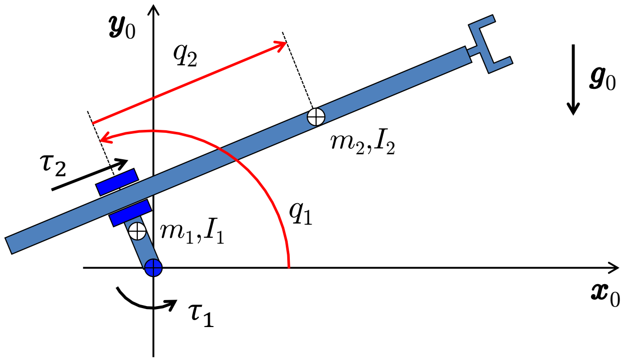

Figure 1: An RP planar robot with an offset at the base.

Exercise 2

For the RP robot of Fig. 1, consider the joint trajectory

$$\boldsymbol{q}(t) = \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} = \begin{pmatrix} a + b\left(1 - \cos\dfrac{\pi t}{T}\right) \\ k \end{pmatrix} \qquad t \in [0, T], \tag{1}$$

with positive values a, b, k and T. Compute the value of the integral of the generalized momentum $\boldsymbol{p} = \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$ when the robot moves along this trajectory, namely the vector

$$\boldsymbol{h} = \int_0^T \boldsymbol{p}(t)\, dt.$$

Exercise 3

For a generic robot with n joints, assume that the velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^n$ is the control input (kinematic control). At a given configuration $\boldsymbol{q}$, determine the optimal command $\dot{\boldsymbol{q}}^*$ that solves the following optimization problem:

$$\min_{\dot{\boldsymbol{q}}} \ \boldsymbol{p}^T \dot{\boldsymbol{q}} \quad \text{s.t.} \quad \boldsymbol{p}_a = \boldsymbol{c},$$

being $\boldsymbol{p} \in \mathbb{R}^n$ the generalized momentum, $\boldsymbol{p}_a \in \mathbb{R}^{n_a}$ the vector of its first $n_a < n$ components, and $\boldsymbol{c} \in \mathbb{R}^{n_a}$ a constant vector with all positive components. Provide then the explicit expression of the solution $\dot{\boldsymbol{q}}^*$ of this problem for the case of the RP robot in Fig. 1.

Exercise 4

Figure 2 shows the RP robot of Exercise 1 in contact with a vertical wall that is assumed to be compliant. The contact point is at (x_E, y_E). Design a regulation control law for $\boldsymbol{\tau} \in \mathbb{R}^2$ such that the robot-environment system behaves during the transient as

$$M_d\, \ddot{x} + D_d\, \dot{x} + K_d\, (x - x_E) = F_d \tag{2}$$

$$\ddot{y} + K_D\, \dot{y} + K_P\, (y - y_E) = 0, \tag{3}$$

in terms of coordinates (x, y) of the end-effector. The target parameters M_d, D_d, K_d, K_D, and K_P are all chosen positive, while $F_d > 0$ is the desired contact force to be applied to the wall. Give the expressions of all the terms needed in the control law and determine the final equilibrium position of the end-effector.

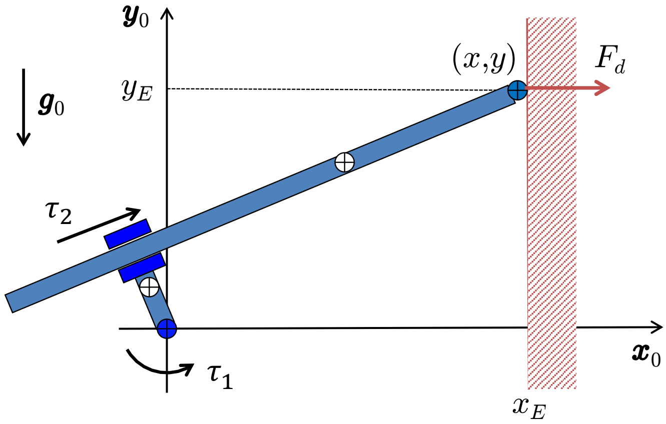

Figure 2: The RP robot in contact with a compliant wall.

Exercise 5

Consider once again the robot in Fig. 1, moving now on a horizontal plane. For the trajectory specified in eq. (1), provide the expression of the torque $\boldsymbol{\tau}(t)$ solving the inverse dynamics problem. Moreover:

- find the minimum motion time $T = T^*$ of the trajectory such that the torque τ_1 at the first joint satisfies the bound $|\tau_1(t)| \leq \tau_{max,1}$, for all $t \in [0, T^*]$;
- accordingly, compute the value $\tau_2(\bar{t})$ of the force at the second joint in correspondence to the time instant(s) $\bar{t}$ at which the torque of the first joint saturates, i.e., when $|\tau_1(\bar{t})| = \tau_{max,1}$.

[210 minutes (3.5 hours); open books]

Solution

September 19, 2024

Exercise 1

First note that the joint variable q_1 defined in Fig. 1 is the one of a DH convention. In addition to the parameters defined in the figure, let $d_{c1} > 0$ be the distance of the center of mass of link 1 from the first joint axis and $l_1 > 0$ the distance between the two joint axes (i.e., the DH parameter a_1). In the present case of a planar robot, the computation of the kinetic energy T and, from there, of the robot inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ are quite standard:

$$T_1 = \frac{1}{2}\,(I_1 + m_1 d_{c1}^2)\dot{q}_1^2$$

$$\boldsymbol{p}_{c2} = l_1 \begin{pmatrix} c_1 \\ s_1 \end{pmatrix} + q_2 \begin{pmatrix} s_1 \\ -c_1 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \begin{pmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{pmatrix} \begin{pmatrix} q_2 \\ l_1 \end{pmatrix} \dot{q}_1 + \begin{pmatrix} s_1 \\ -c_1 \end{pmatrix} \dot{q}_2$$

$$T_2 = \frac{1}{2}\,I_2\dot{q}_1^2 + \frac{1}{2}\,m_2\|\boldsymbol{v}_{c2}\|^2 = \frac{1}{2}\,I_2\dot{q}_1^2 + \frac{1}{2}\,m_2\left((l_1^2+q_2^2)\dot{q}_1^2 + \dot{q}_2^2 - 2l_1\dot{q}_1\dot{q}_2\right)$$

$$T = T_1 + T_2 = \frac{1}{2}\,\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} \quad \Rightarrow \quad \boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} I_1 + m_1 d_{c1}^2 + I_2 + m_2(l_1^2+q_2^2) & -m_2 l_1 \\ -m_2 l_1 & m_2 \end{pmatrix}.$$

From this, the components of the Coriolis and centrifugal vector $\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}})$ are computed using the matrices of Christoffel symbols:

$$\boldsymbol{C}_1(\boldsymbol{q}) = \frac{1}{2}\left\{\begin{pmatrix} 0 & 2m_2q_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 2m_2q_2 & 0 \end{pmatrix} - \boldsymbol{O}\right\} = \begin{pmatrix} 0 & m_2q_2 \\ m_2q_2 & 0 \end{pmatrix}$$

$$\Rightarrow \quad c_1(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T\boldsymbol{C}_1(\boldsymbol{q})\dot{\boldsymbol{q}} = 2m_2q_2\dot{q}_1\dot{q}_2$$

$$\boldsymbol{C}_2(\boldsymbol{q}) = \frac{1}{2}\left\{\boldsymbol{O} + \boldsymbol{O}^T - \begin{pmatrix} 2m_2q_2 & 0 \\ 0 & 0 \end{pmatrix}\right\} = \begin{pmatrix} -m_2q_2 & 0 \\ 0 & 0 \end{pmatrix} \;\Rightarrow\; c_2(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T\boldsymbol{C}_2(\boldsymbol{q})\dot{\boldsymbol{q}} = -m_2q_2\dot{q}_1^2.$$

Similarly, for the potential energy U and gravity vector $\boldsymbol{q}(\boldsymbol{q})$:

$$U_1 = m_1 g_0 d_{c1} s_1 \qquad U_2 = m_2 g_0\,(l_1 s_1 - q_2 c_1) \qquad \Rightarrow \qquad U = U_1 + U_2$$

$$\Rightarrow \quad \boldsymbol{g}(\boldsymbol{q}) = \frac{\partial U}{\partial \boldsymbol{q}} = g_0 \begin{pmatrix} (m_1 d_{c1} + m_2 l_1)c_1 + m_2 q_2 s_1 \\ -m_2 c_1 \end{pmatrix}.$$

Finally, introducing the $r = 3$ dynamic coefficients

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} I_1 + m_1 d_{c1}^2 + I_2 + m_2 l_1^2 \\ m_2 \\ m_1 d_{c1} + m_2 l_1 \end{pmatrix},$$

the complete dynamic model is linearly parametrized as

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{Y}(\boldsymbol{q}.\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})\,\boldsymbol{a} = \boldsymbol{\tau},$$

with the 2×3 regressor matrix $\boldsymbol{Y}$ of the linear parametrization given by

$$\boldsymbol{Y} = \begin{pmatrix} \ddot{q}_1 & -l_1\ddot{q}_2 + 2q_2\dot{q}_1\dot{q}_2 + g_0q_2s_1 & g_0c_1 \\ 0 & \ddot{q}_2 - l_1\ddot{q}_1 - g_0c_1 & 0 \end{pmatrix},$$

where $g_0 = 9.81$ m/s^2 and the kinematic length $l_1 > 0$ are assumed to be known.

The number of dynamic coefficients is indeed minimal, unless the first link is balanced around the axis of its rotation, i.e., of joint 1, in which case $d_{c1} = 0$; the third dynamic coefficient becomes then proportional to $a_2 = m_2$ (by the known factor l_1) and is thus no longer needed.

Exercise 2

Along the trajectory in eq. (1), since $q_2 = k$ the inertia matrix of the RP robot in Fig. 1 becomes constant:

$$\bar{\boldsymbol{M}} = \begin{pmatrix} I_1 + m_1 d_{c1}^2 + I_2 + m_2(l_1^2 + k^2) & -m_2 l_1 \\ -m_2 l_1 & m_2 \end{pmatrix}.$$

Therefore, the integral of the generalized momentum is easily computed as

$$\boldsymbol{h} = \int_0^T \boldsymbol{p}(t)\,dt = \bar{\boldsymbol{M}} \int_0^T \dot{\boldsymbol{q}}(t)\,dt = \bar{\boldsymbol{M}} \int_0^T \frac{d\boldsymbol{q}}{dt}\,dt = \bar{\boldsymbol{M}} \int_{\boldsymbol{q}(0)}^{\boldsymbol{q}(T)} d\boldsymbol{q} = \bar{\boldsymbol{M}} \left(\boldsymbol{q}(T) - \boldsymbol{q}(0)\right).$$

Being from (1)

$$\boldsymbol{q}(0) = \begin{pmatrix} a \\ k \end{pmatrix} \qquad \boldsymbol{q}(T) = \begin{pmatrix} a + 2b \\ k \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{q}(T) - \boldsymbol{q}(0) = \begin{pmatrix} 2b \\ 0 \end{pmatrix},$$

we obtain

$$\boldsymbol{h} = 2b \begin{pmatrix} I_1 + m_1 d_{c1}^2 + I_2 + m_2(l_1^2 + k^2) \\ -m_2 l_1 \end{pmatrix}.$$

Exercise 3

Partition the robot inertia matrix by rows and columns as

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{M}_a(\boldsymbol{q}) \\ \boldsymbol{M}_b(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{M}_{aa}(\boldsymbol{q}) & \boldsymbol{M}_{ab}(\boldsymbol{q}) \\ \boldsymbol{M}_{ba}(\boldsymbol{q}) & \boldsymbol{M}_{bb}(\boldsymbol{q}) \end{pmatrix},$$

where $\boldsymbol{M}_{ba} = \boldsymbol{M}_{ab}$, with the dimensions

$$\boldsymbol{M}_a : \; n_a \times n \qquad \boldsymbol{M}_b : \; (n - n_a) \times n \qquad \boldsymbol{M}_a : \; n_a \times n_a.$$

Note also that matrix $\boldsymbol{M}_a$ has always full (row) rank n_a (since all rows of $\boldsymbol{M}$ are linearly independent) and the square matrix $\boldsymbol{M}_{aa}$ is nonsingular (as a diagonal block of the inertia matrix).

With the above in mind, the problem is a regular LQ optimization as encountered in the resolution of redundancy:

$$\min_{\dot{\boldsymbol{q}}} \; \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q}) \boldsymbol{q} \quad \text{s.t.} \quad \boldsymbol{M}_a(\boldsymbol{q}) \dot{\boldsymbol{q}} = \boldsymbol{c},$$

with the quadratic objective having a weighting matrix $\boldsymbol{M} > 0$ (and actually being equal to twice the robot kinetic energy) and the $\boldsymbol{M}_a$ playing the role of a 'Jacobian' in the linear equality constraint. Thus, the solution has the form of a weighted pseudoinverse:

$$\dot{\boldsymbol{q}}^* = \boldsymbol{M}^{-1}(\boldsymbol{q}) \boldsymbol{M}_a^T(\boldsymbol{q}) \left(\boldsymbol{M}_a(\boldsymbol{q}) \boldsymbol{M}^{-1}(\boldsymbol{q}) \boldsymbol{M}_a^T(\boldsymbol{q}) \right)^{-1} \boldsymbol{c}. \tag{4}$$

Moreover, it is easy to see that simplifications occur in this case since the matrix in the constraint is part of the weighting matrix of the problem. In fact, from the identity

$$\boldsymbol{I} = \boldsymbol{M}\boldsymbol{M}^{-1} = \begin{pmatrix} \boldsymbol{M}_a \\ \boldsymbol{M}_b \end{pmatrix} \boldsymbol{M}^{-1} = \begin{pmatrix} \boldsymbol{I}_a & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I}_b \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{M}_a \boldsymbol{M}^{-1} = \begin{pmatrix} \boldsymbol{I}_a & \boldsymbol{O} \end{pmatrix},$$

it also follows

$$
\boldsymbol{M}^{-1}\boldsymbol{M}_a^T = \begin{pmatrix} \boldsymbol{I}_a \\ \boldsymbol{O} \end{pmatrix} \qquad \boldsymbol{M}_a^T\boldsymbol{M}^{-1}\boldsymbol{M}_a = \boldsymbol{M}_{aa}.
$$

Substituting these in (4) yields finally

$$
\dot{\boldsymbol{q}}^* = \begin{pmatrix} \dot{\boldsymbol{q}}_a^* \\ \dot{\boldsymbol{q}}_b^* \end{pmatrix} = \begin{pmatrix} \boldsymbol{M}_{aa}^{-1}(\boldsymbol{q})\,\boldsymbol{c} \\ \boldsymbol{0} \end{pmatrix}.
$$

The solution is easy to be interpreted: in order to minimize the robot kinetic energy when some components of the generalized momentum are kept constant, the homologous components of the joint velocity, i.e., $\dot{\boldsymbol{q}}_a$, are used to maintain the constraint while the remaining joint velocity components, i.e., $\dot{\boldsymbol{q}}_b$, are set to zero. Note that if $\boldsymbol{c} = \boldsymbol{0}$, the solution is simply $\dot{\boldsymbol{q}}^* = \boldsymbol{0}$ (the robot does not move). With a trivial modification, the result applies also to the case when an arbitrary subset of components of $\boldsymbol{p}$ is kept constant — not necessarily the first n_a components.

Applying the solution to the RP robot in Fig. 1 gives

$$
\dot{\boldsymbol{q}}^* = \begin{pmatrix} \dfrac{c}{I_1 + m_1 d_{c1}^2 + I_2 + m_2(l_1^2 + q_2^2)} \\ 0 \end{pmatrix}.
$$

Exercise 4

Despite the robot task is a Cartesian regulation problem (of the hybrid type), i.e., to keep a constant position while applying a constant force with the end-effector, the desired linear and decoupled dynamics of the closed-loop system expressed by eqs. (2) and (3) requires the resort to a feedback linearization approach.

For this, we also need to define the Jacobian matrix for the position of the robot end-effector. Let $l_2 > 0$ be the distance from the center of mass of the second link to the end-effector. From

$$
\boldsymbol{p}_{ee} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} l_1 c_1 + (q_2 + l_2) s_1 \\ l_1 s_1 - (q_2 + l_2) c_1 \end{pmatrix}
$$

we obtain

$$
\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}_{ee}}{\partial \boldsymbol{q}} = \begin{pmatrix} -l_1 s_1 + (q_2 + l_2) c_1 & s_1 \\ l_1 c_1 + (q_2 + l_2) s_1 & -c_1 \end{pmatrix}.
$$

The determinant of this matrix is $\det \boldsymbol{J} = -(q_2 + l_2)$.

The dynamic model of the robot in contact is then

$$
\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} + \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{F}, \tag{5}
$$

where $\boldsymbol{F} \in \mathbb{R}^2$ is a generic force applied from the environment to the robot end-effector, while the terms $\boldsymbol{M}$, $\boldsymbol{c}$, and $\boldsymbol{g}$ have already been defined in the solution of Exercise 1. The control law that achieves feedback linearization in the Cartesian space is

$$
\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}^{-1}(\boldsymbol{q})\left(\boldsymbol{a} - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right) + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{F}, \tag{6}
$$

where we assumed to be out of singularities, i.e., $q_2 \neq -l_2$. The only term still missing is the derivative of the Jacobian matrix, namely

$$
\dot{\boldsymbol{J}}(\boldsymbol{q}) = \begin{pmatrix} -(l_1 c_1 + (q_2 + l_2) s_1)\dot{q}_1 + c_1 \dot{q}_2 & c_1 \dot{q}_1 \\ (-l_1 s_1 + (q_2 + l_2) c_1)\dot{q}_1 + s_1 \dot{q}_2 & s_1 \dot{q}_1 \end{pmatrix},
$$

which appears in the end-effector acceleration

$$\ddot{\boldsymbol{p}}_{ee} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}.$$

Using the law (6) in (5) gives

$$\ddot{\boldsymbol{p}}_{ee} = \boldsymbol{a} \qquad \text{or} \qquad \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} a_x \\ a_y \end{pmatrix},$$

thus achieving exact linearization and input-output decoupling. The synthesis of the two components of the acceleration command $\boldsymbol{a}$ is completed as

$$a_x = \frac{1}{M_d}\left(K_d\,(x_E - x) - D_d\,\dot{x} + F_d\right) \tag{7}$$

$$a_y = K_P\,(y_E - y) - K_D\,\dot{y}, \tag{8}$$

giving the final desired dynamic behavior. At the equilibrium ($\dot{x} = \dot{y} = 0$, $\ddot{x} = \ddot{y} = 0$), from eqs. (2) and (3) one has

$$\bar{x} = x_E + \frac{1}{K_d}F_d \qquad \bar{y} = y_E \qquad \bar{F}_x = K_d\,(\bar{x} - x_E) = F_d,$$

i.e., the robot end-effector penetrates ($\bar{x} > x_E$) by a (small) amount horizontally into the wall, so as to realize the desired contact force F_d, whereas its vertical position is kept at the desired height y_E of the contact point.

Note that the control law can be interpreted as a hybrid force-position scheme in which the force loop is designed mimicking an impedance law, with the desired contact force F_d in place of the measured one (i.e., F_x) on the right-hand side of eq. (2).

Exercise 5

The first and second time derivatives of the trajectory (1) are

$$\dot{\boldsymbol{q}}(t) = \begin{pmatrix} \dfrac{b\pi}{T}\sin\dfrac{\pi t}{T} \\ 0 \end{pmatrix} \qquad \ddot{\boldsymbol{q}}(t) = \begin{pmatrix} \dfrac{b\pi^2}{T^2}\cos\dfrac{\pi t}{T} \\ 0 \end{pmatrix}.$$

Setting $\boldsymbol{g} \equiv \boldsymbol{0}$ and substituting the trajectory in the dynamic model yields for the inverse dynamics

$$\boldsymbol{\tau}(t) = \begin{pmatrix} \tau_1(t) \\ \tau_2(t) \end{pmatrix} = \begin{pmatrix} \dfrac{(a_1 + m_2k^2)\,b\pi^2}{T^2}\cos\dfrac{\pi t}{T} \\ -\dfrac{m_2b\pi^2}{T^2}\left(l_1\cos\dfrac{\pi t}{T} + k\sin^2\dfrac{\pi t}{T}\right) \end{pmatrix} \qquad t \in [0, T],$$

with the already defined dynamic coedfficient $a_1 = I_1 + m_1d_{c1}^2 + I_2 + m_2l_1^2$.

The maximum absolute value of the torque at the first joint is reached at the initial and final instants ($\bar{t} = \{0, T\}$):

$$|\tau_1(0)| = |\tau_1(T)| = \frac{(a_1 + m_2k^2)\,b\pi^2}{T^2} \leq \tau_{max,1}.$$

Thus, the minimum feasible motion time is

$$T^* = \sqrt{\frac{(a_1 + m_2k^2)\,b\pi^2}{\tau_{max,1}}}.$$

Accordingly, the force at the second joint in the two instants $\bar{t} = 0$ and $\bar{t} = T$ will be:

$$\tau_2(0) = -\frac{m_2 l_1}{a_1 + m_2 k^2}\,\tau_{max,1} \qquad\qquad \tau_2(T) = -\tau_2(0).$$

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