Control Systems

Time response

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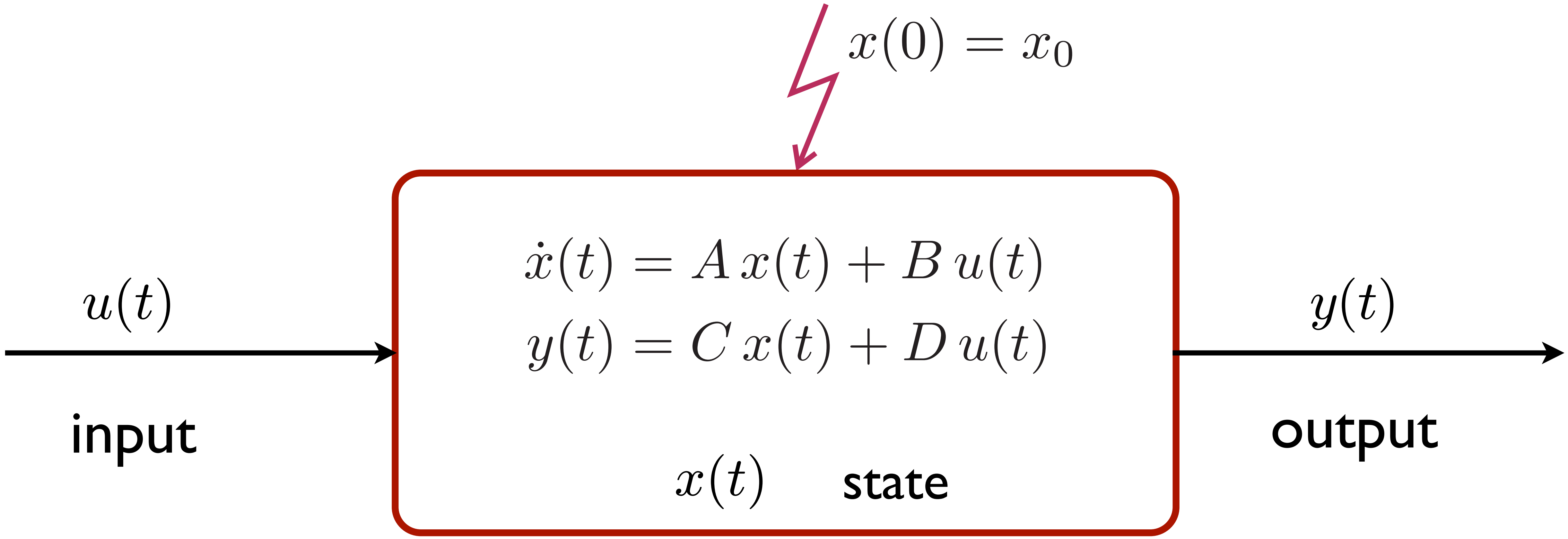
outline

- zero-state solution
- matrix exponential
- total response (sum of zero-state and zero-input responses)
- Dirac impulse
- impulse response
- change of coordinates (state)

system



$x(0) = x_0$

$u(t)$

input

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$
$$y(t) = C\,x(t) + D\,u(t)$$

$x(t)$ state

$y(t)$

output

Linear Time Invariant (LTI)
dynamical system
(in Continuous Time)

$$x \in \mathbb{R}^n \quad u \in \mathbb{R}^m \quad y \in \mathbb{R}^p$$

system representation

implicit representation

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(0) = x_0$$

$$\dot{v} = \frac{F}{m} \qquad v(0) = v_0$$

↑

example

↓

$$v(t) = v_0 + \frac{1}{m}\int_0^t F(\tau)d\tau$$

explicit representation (**solution**)

$$x(t) = \ldots$$

- we want to study the solution of the set of differential equations in order to have a **qualitative** knowledge of the system motion
- we need the general expression of the solution
  - first we look at the solution when no input is applied (**zero-input response**)
  - then we add the contribution due to the input only (**zero-state response**)

solution: zero-input response

zero-input response (i.e. with $u(t) = 0$) $\qquad \dot{x}(t) = Ax(t) + B\cancel{u(t)}$

- scalar case $\quad \dot{x} = ax \qquad x(0) = x_0 \quad \longrightarrow \quad x(t) = e^{at}x_0$
- matrix case $\quad \dot{x} = Ax \qquad x(0) = x_0 \quad \longrightarrow \quad x(t) = e^{At}x_0$

dimensions $\quad n \times 1 \quad \downarrow \quad n \times 1$

$n \times n$

what is e^{At} ?

definition

$$\boxed{e^{At} = I + At + A^2\frac{t^2}{2!} + \cdots = \sum_{k=0}^{\infty} A^k \frac{t^k}{k!}}$$

check

$$\dot{x} = \frac{d}{dt}\left(e^{At}x_0\right) = \cdots = Ax$$

matrix exponential

definition

$$\boxed{e^{At} = I + At + A^2\frac{t^2}{2!} + \cdots = \sum_{k=0}^{\infty} A^k \frac{t^k}{k!}}$$

properties

$$\left. e^{At} \right|_{t=0} = I$$

consistency

$$e^{At_1} \cdot e^{At_2} = e^{A(t_1+t_2)}$$

composition

$$e^{A_1 t} \cdot e^{A_2 t} \neq e^{(A_1+A_2)t}$$

$$\underset{\text{equality holds iff}}{=} \iff A_1A_2 = A_2A_1$$

$$\left(e^{At}\right)^{-1} = e^{-At}$$

$$\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$$

solution: zero-input response

the exponential matrix propagates the initial condition into state at time t

$$x_0 = x(0) \xrightarrow[\text{propagation}]{e^{At}} x(t)$$

more in general it propagates the state t seconds forward in time

$$x(\tau) \xrightarrow{\qquad\qquad} x(\tau + t) = e^{At} x(\tau)$$

since $x(\tau + t) = e^{A(\tau+t)} x(0) = e^{At} e^{A\tau} x(0) = e^{At} x(\tau)$

curiosity: Euler approximation

$$x(\tau + t) \approx x(\tau) + t\dot{x}(\tau) = (I + At)x(\tau)$$

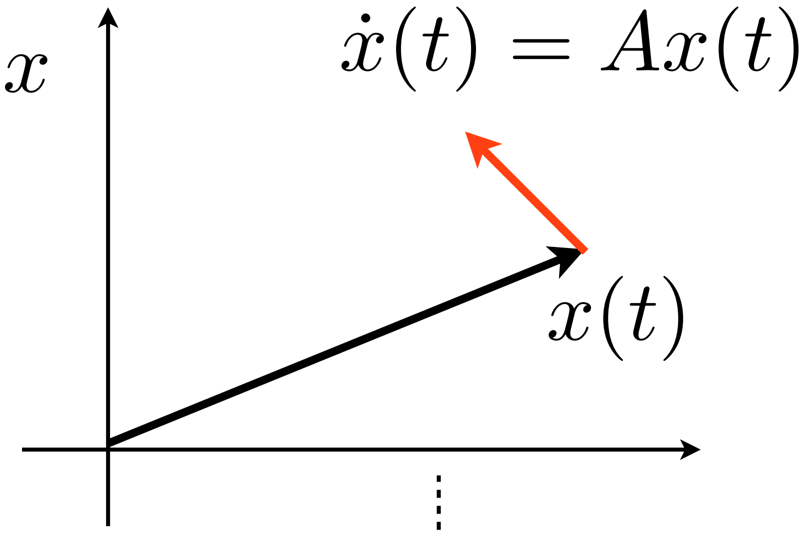
exact solution

$$x(\tau + t) = e^{At} x(\tau) = (\boxed{I + At} + A^2t^2/2! + \ldots)x(\tau)$$

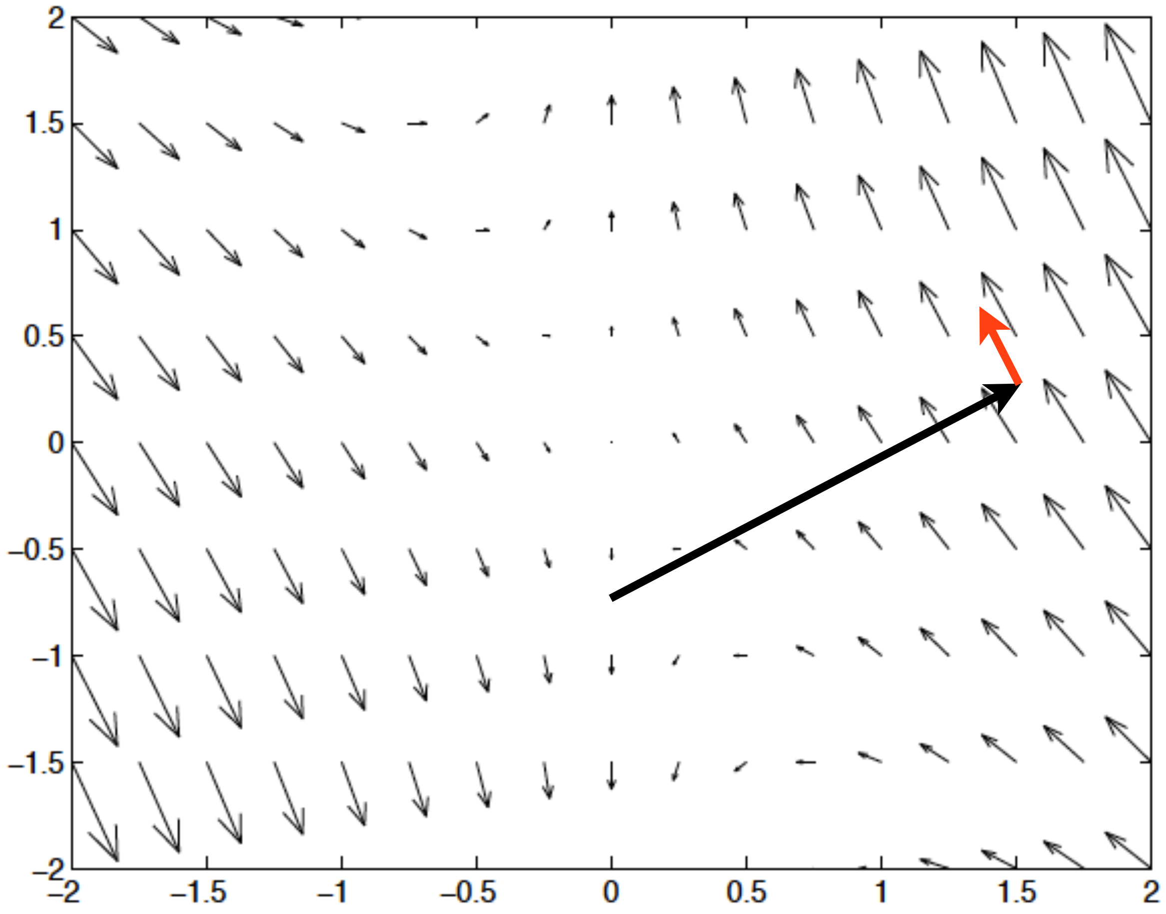
solution: zero-input response

phase plane $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ (note that $n = 2$, hence it's a plane)

it is possible to represent the **vector field** $\dot{x} = Ax$



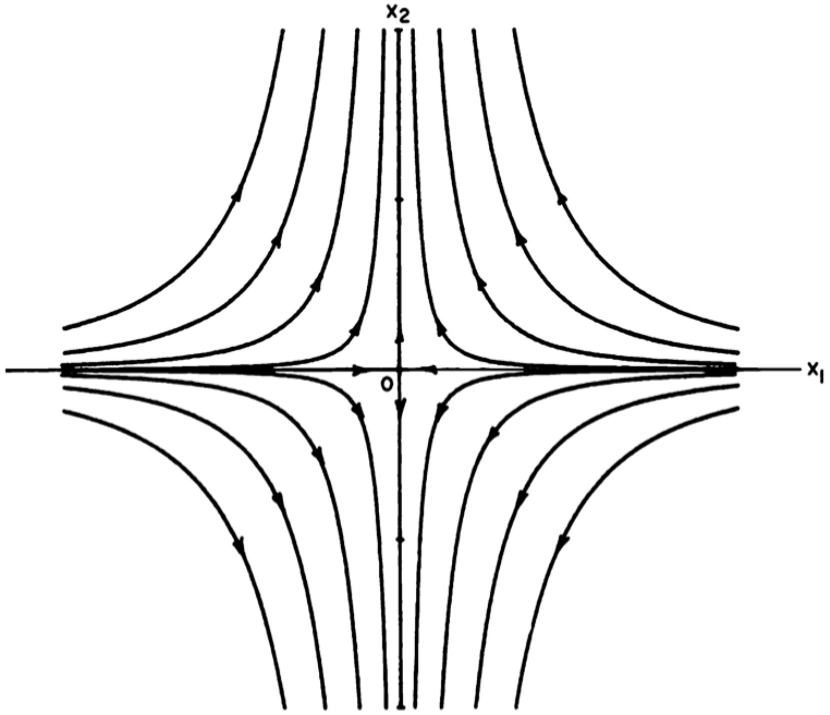
vector field



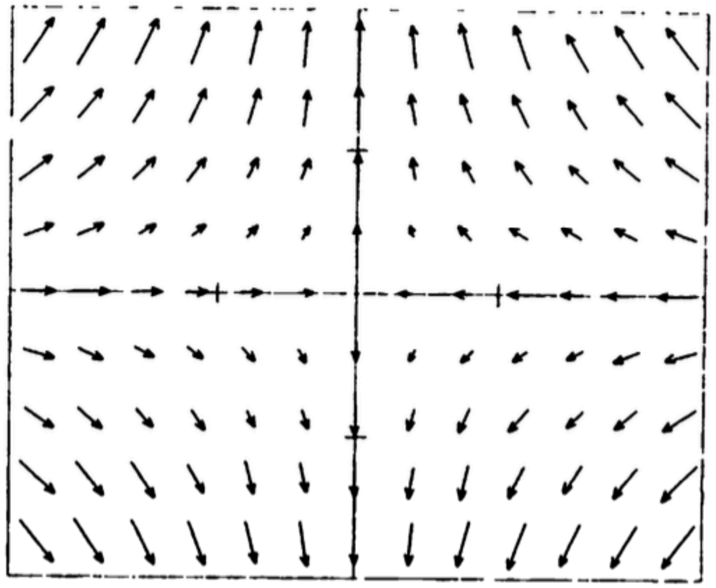
example $\dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} x(t)$

and build a **phase portrait** (geometric representation of the trajectories)

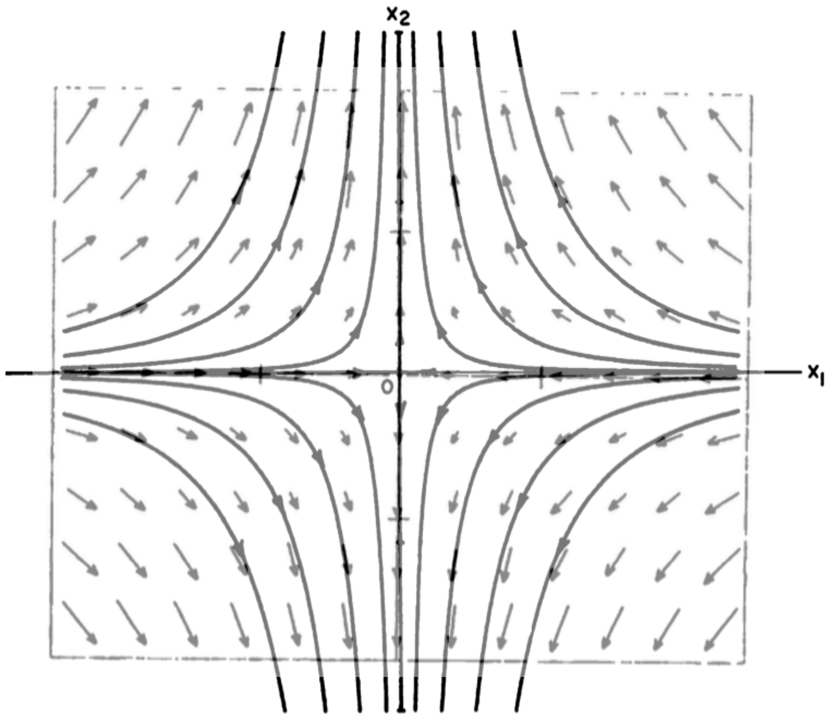
for mechanical systems (typically second order systems) the phase plane has coordinates position and velocity



phase plane
state trajectories



vector field

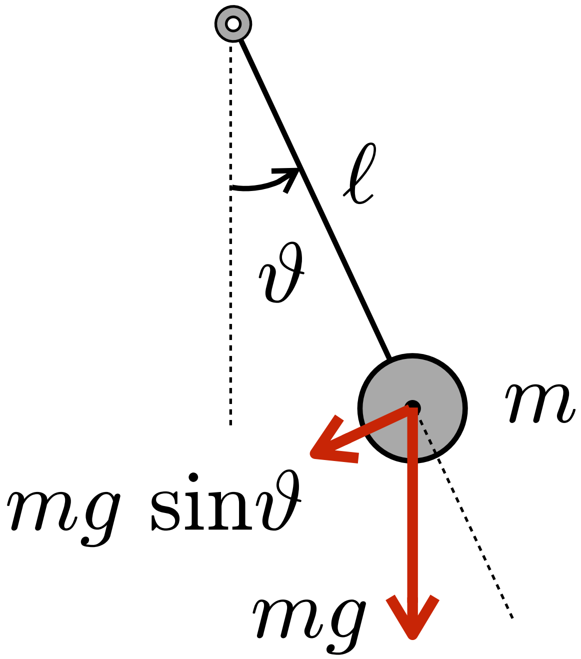


together

solution on phase plane: pendulum example

modeling hypothesis

- rod has no weight
- mass m concentrated at the tip



moment of inertia is $I = m\ell^2$

equation of motion $\quad I\,\ddot{\vartheta} = -\ell\, m\, g\ \sin\vartheta$

Pendulum with no damping (**nonlinear** differential equation) $\quad \ddot{\vartheta}(t) + \dfrac{g}{\ell}\sin\vartheta(t) = 0$

in state space form, choosing the state as $\quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \vartheta \\ \dot{\vartheta} \end{bmatrix}$

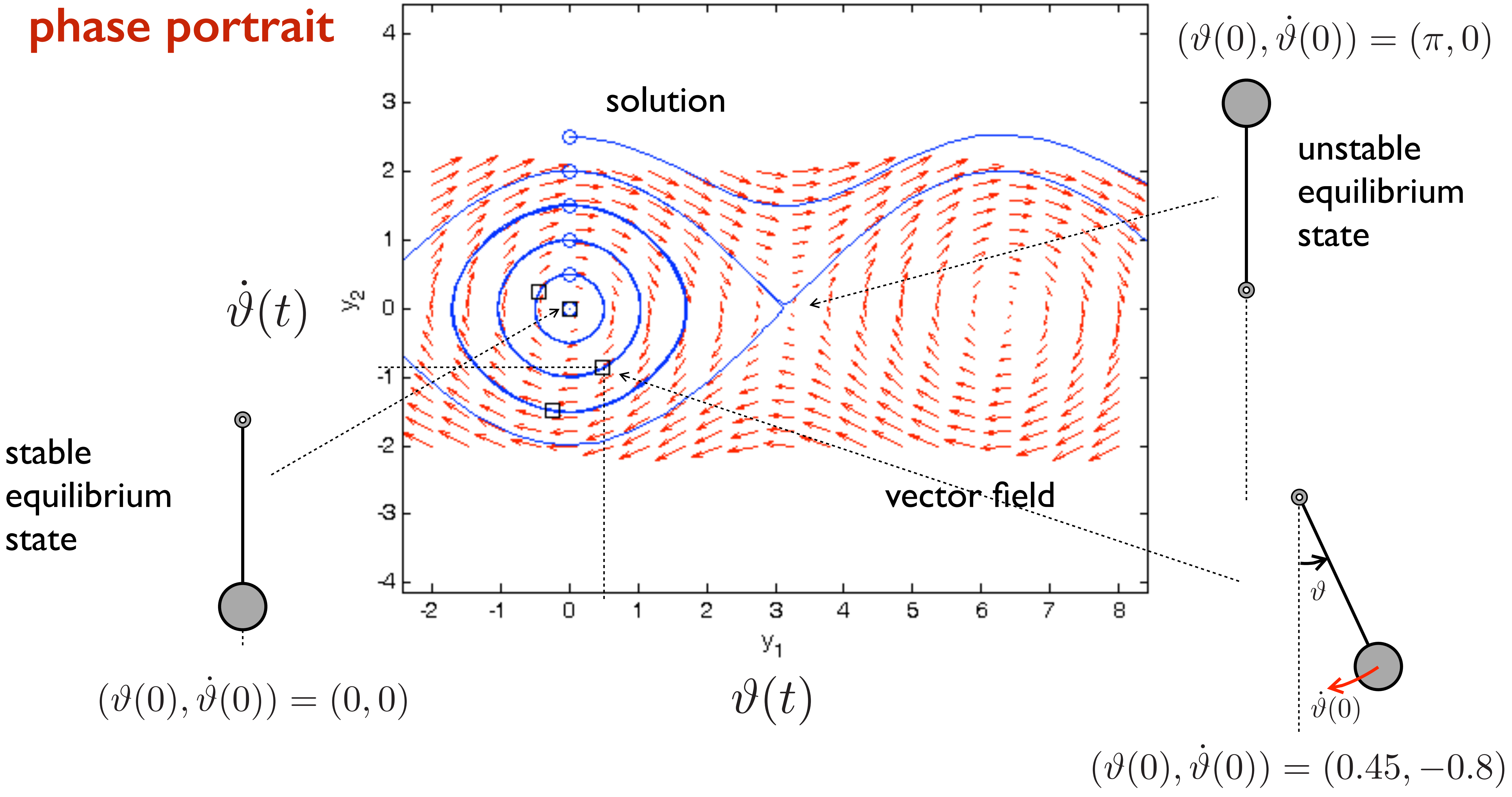
gives
$$\dot{x} = \begin{bmatrix} x_2 \\ -\frac{g}{\ell}\sin x_1 \end{bmatrix} = f(x)$$

solution on phase plane: pendulum example

Pendulum with no damping
(**nonlinear** differential equation)

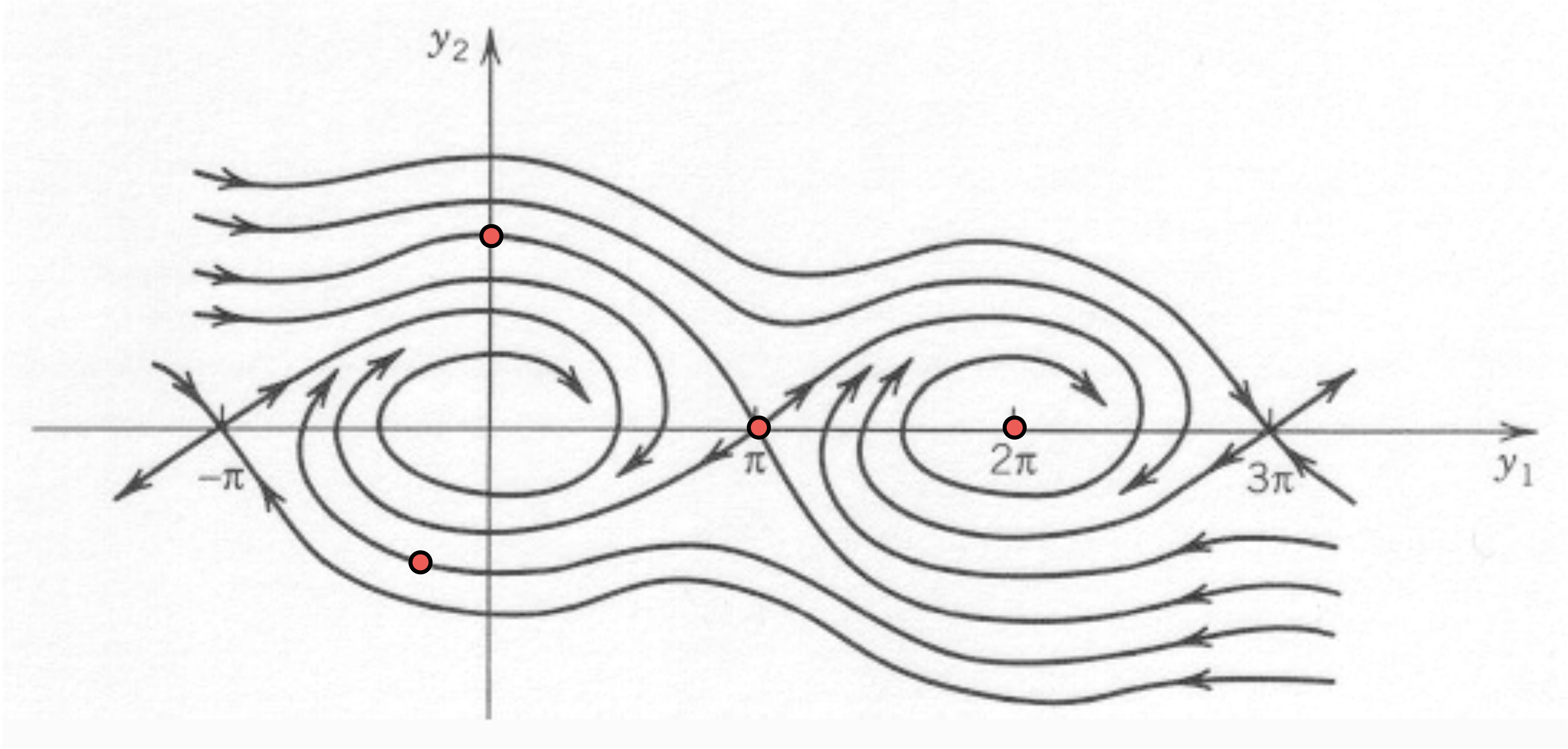
$$\ddot{\vartheta}(t) + \frac{g}{\ell}\sin\vartheta(t) = 0$$

phase portrait



solution on phase plane: damped pendulum

more in the stability section



interpret the different motions starting from the initial conditions ●

solution: total response - general case

$$\dot{x} = Ax + Bu$$

$$\dot{x} - Ax = Bu$$

$$e^{-At}\dot{x} - e^{-At}Ax = e^{-At}Bu$$

$$\frac{d}{dt}\left(e^{-At}x(t)\right) = e^{-At}Bu(t)$$

$$\left(e^{-A\tau}x(\tau)\right)\Big|_{\tau=0}^{t} = \int_0^t e^{-A\tau}Bu(\tau)d\tau$$

$$e^{-At}x(t) - e^{-A0}x(0) = \int_0^t e^{-A\tau}Bu(\tau)d\tau$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

solution: total response

scalar case

$\dot{x} = ax + bu$ solution is $x(t) = e^{at}x_0 + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau$

check using the Leibniz integral rule

$$\frac{d}{dt}\int_0^t f(t,\tau)d\tau = \int_0^t \frac{d}{dt}f(t,\tau)d\tau + f(t,\tau)\big|_{\tau=t}$$

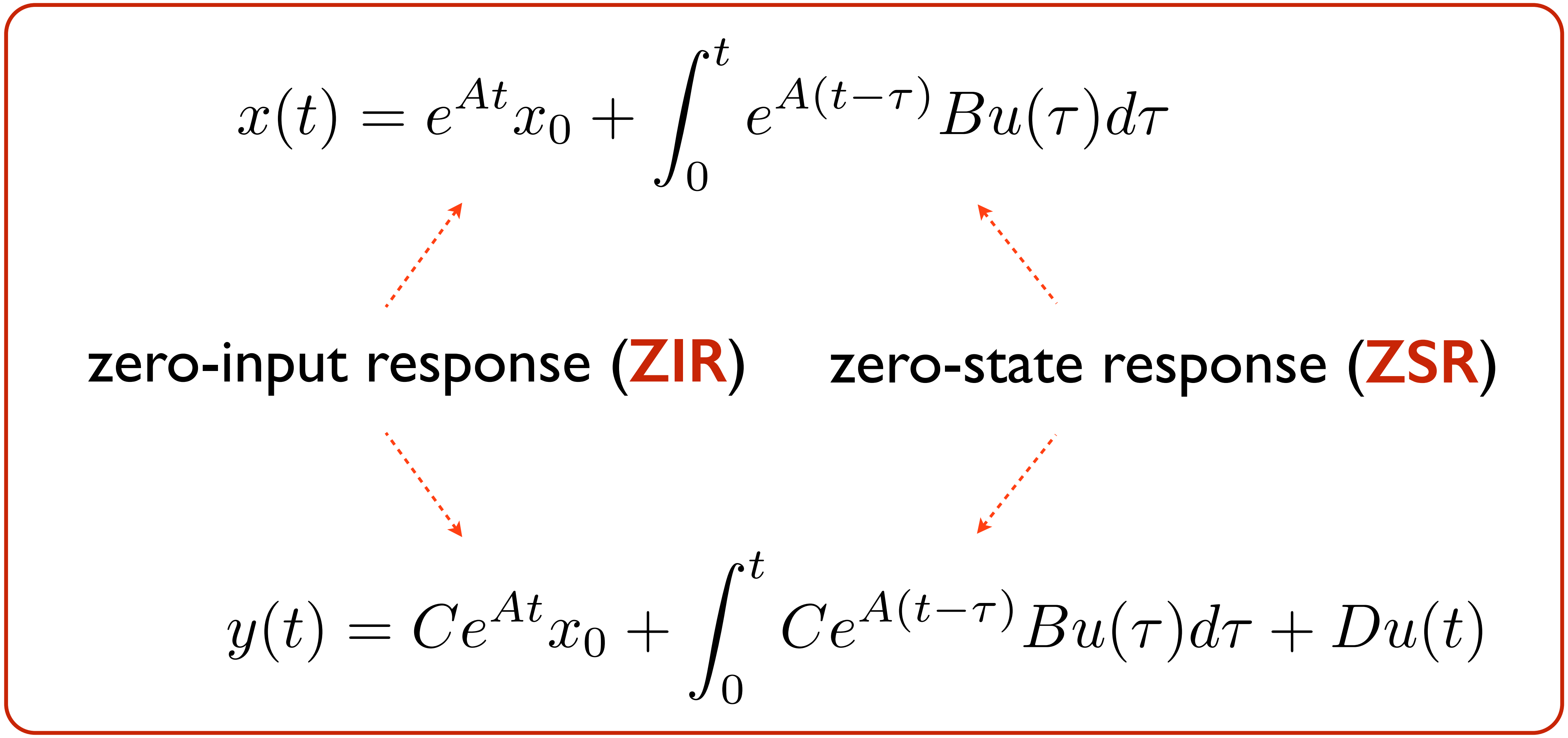
$$\dot{x} = ae^{at}x_0 + \int_0^t ae^{a(t-\tau)}bu(\tau)d\tau + bu(t) = ax + bu \qquad \blacksquare$$

example: see the point mass $v(t) = v_0 + \frac{1}{m}\int_0^t F(\tau)d\tau$

definition: convolution integral $f(t) \star g(t) = \int_0^t f(t-\tau)g(\tau)d\tau$

solution: total response

matrix case



state

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

zero-input response (**ZIR**)   zero-state response (**ZSR**)

output

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

e^{At} state-transition matrix

Ce^{At} output-transition matrix

N.B. product is **not** commutative for matrices

solution: total response

total response = zero-input response + zero-state response

$x(0) \neq 0$ $\quad$ $x(0) = 0$

$u(t) = 0$ $\quad$ $u(t) \neq 0$

two distinct contributions to the motion of a linear system

- a non-zero initial condition causes motion as well as
- a non-zero input

alternative names

- zero-input response = **free** response/evolution
- zero-state response = **forced** response/evolution

superposition principle

consequence of linearity

from general solution

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

if $(x_{0a}, u_a(t))$ generates $x_a(t)$ and $y_a(t)$

if $(x_{0b}, u_b(t))$ generates $x_b(t)$ and $y_b(t)$

then $(\alpha x_{0a} + \beta x_{0b}, \alpha u_a(t) + \beta u_b(t))$ generates $\alpha x_a(t) + \beta x_b(t)$

and $\alpha y_a(t) + \beta y_b(t)$

only for **same** linear combination

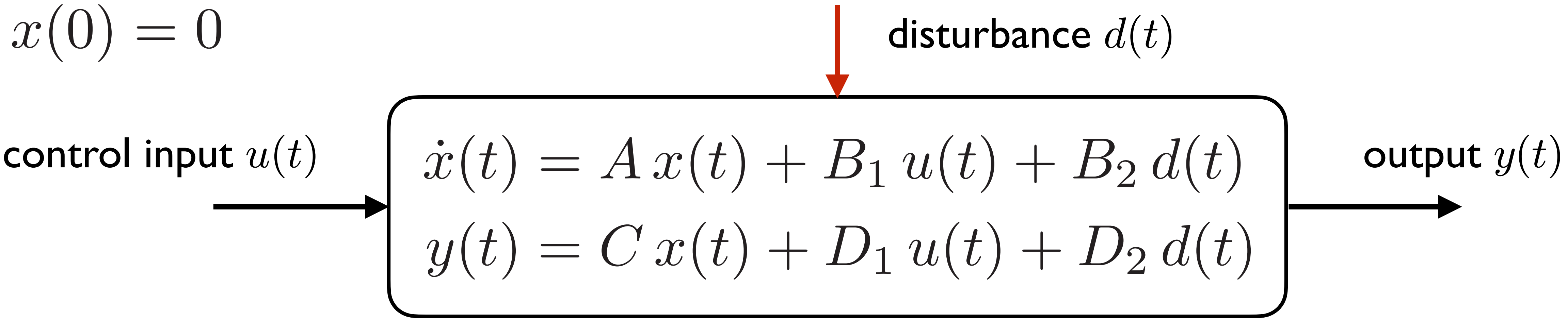
special case ZSR (forced response) $x(0) = 0$

if $u_a(t) \longrightarrow$ ZSR$_a$

$u_b(t) \longrightarrow$ ZSR$_b$

then $\alpha u_a(t) + \beta u_b(t) \longrightarrow \alpha\text{ZSR}_a + \beta\text{ZSR}_b$

superposition principle: special case - example



$x(0) = 0$

disturbance $d(t)$

control input $u(t)$

$$\dot{x}(t) = A\,x(t) + B_1\,u(t) + B_2\,d(t)$$
$$y(t) = C\,x(t) + D_1\,u(t) + D_2\,d(t)$$

output $y(t)$

total output ZSR (forced response) is made up of two contributions

- a first due only to the control input (setting the disturbance to zero)
- a second due only to the disturbance (setting the control input to zero)

$$y(t) = \underset{\substack{\uparrow \\ u(t) \neq 0 \\ d(t) = 0}}{y_{\mathrm{ci}}(t)} + \underset{\substack{\uparrow \\ u(t) = 0 \\ d(t) \neq 0}}{y_{\mathrm{d}}(t)}$$

property frequently used in control design

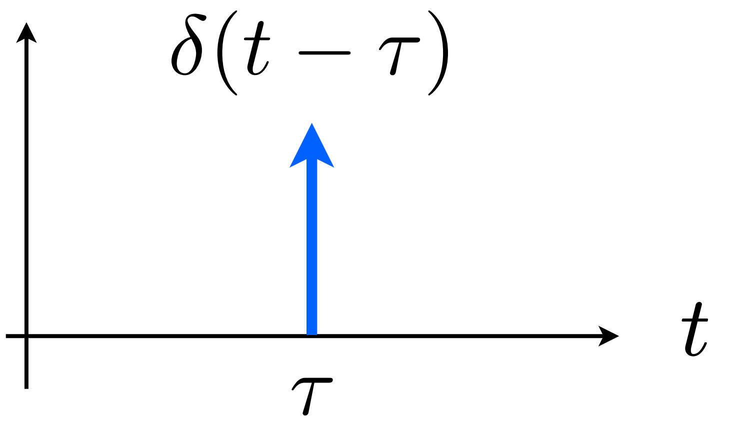
Dirac's delta (impulse)

generalized function

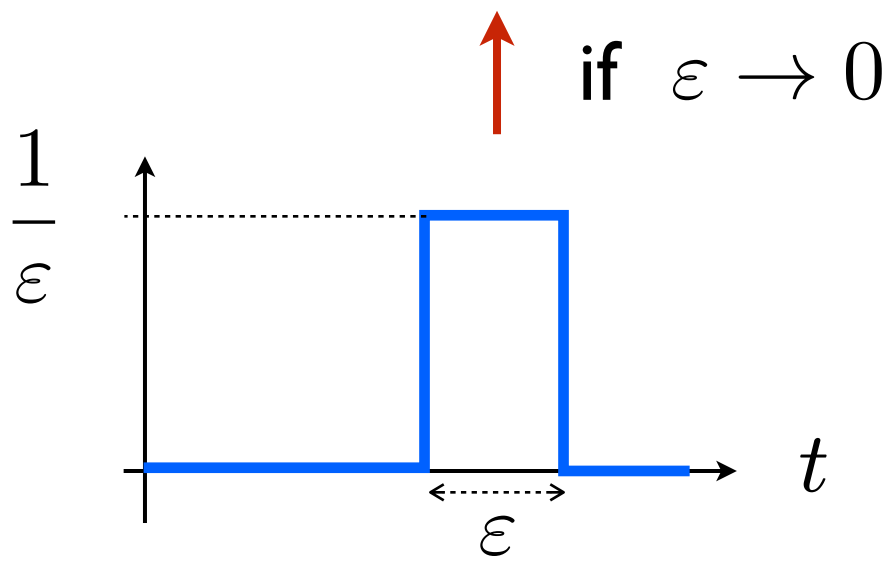
$$\delta(t) = 0 \quad \text{if} \quad t \neq 0$$

$$\int_{-\infty}^{+\infty} \delta(\tau) d\tau = 1$$

impulse
centered in $t = \tau$
(or centered in $\tau = t$)



approximation



properties

$$f(t)\delta(t-\tau) = f(\tau)\delta(t-\tau)$$ (as a function of t)

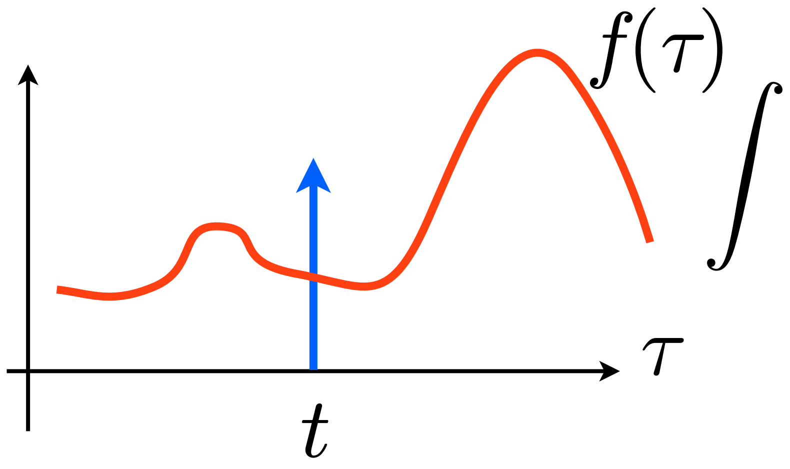
$$f(\tau)\delta(t-\tau) = f(t)\delta(t-\tau)$$ (as a function of τ)

$$\boxed{\int_{-\infty}^{+\infty} f(\tau)\delta(t-\tau)d\tau = f(t)}$$

$$\boxed{\int_{-\infty}^{+\infty} f(t-\tau)\delta(\tau)d\tau = f(t)}$$

sifting
(sampling)
property

(as a function of τ)



Dirac's delta: application

zero-state output response

sifting property

$$\int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \overset{\updownarrow}{=} \int_0^t \left[Ce^{A(t-\tau)}B + D\delta(t-\tau)\right]u(\tau)d\tau$$

rewritten as $\int_0^t W(t-\tau)u(\tau)d\tau$ with $\boxed{W(t) = Ce^{At}B + D\delta(t)}$

for now just a more compact way to rewrite the ZSR, but it can be also given an interesting physical interpretation

$$\int_0^t W(t-\tau)u(\tau)d\tau \xrightarrow{\text{if } u(t)=\delta(t)} \int_0^t W(t-\tau)\delta(\tau)d\tau = W(t)$$

$W(t)$ defined as the (output) **impulse response**
i.e. the response to a specific input, the Dirac impulse

impulse response

for any input $u(t)$, the zero-state output response is a convolution integral of the impulse response $W(t)$ with the input $u(t)$

$$\boxed{\text{output ZSR} = \int_0^t W(t-\tau)u(\tau)d\tau}$$

to the input $u(t)$

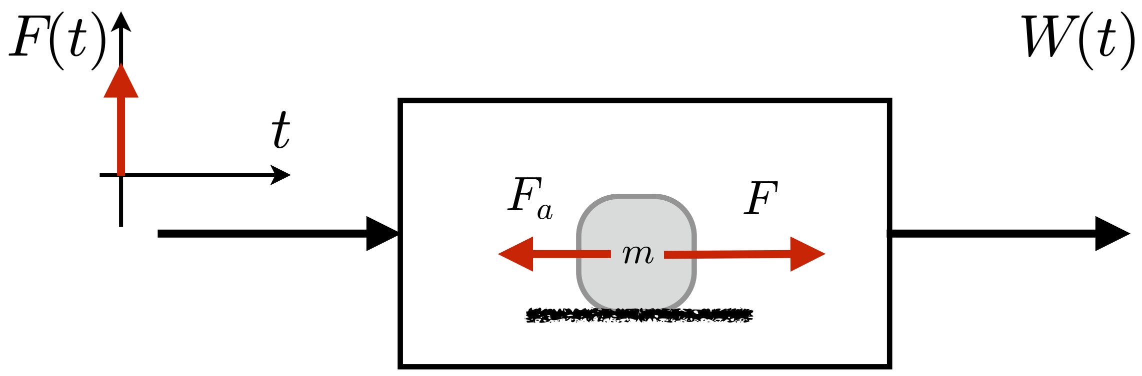
- the knowledge of the sole (output) impulse response $W(t)$ allows us to predict the zero-state (output) response to any input $u(t)$
- a unique experiment aimed at the determination of the impulse response $W(t)$ is, theoretically, sufficient to characterize any zero-state response

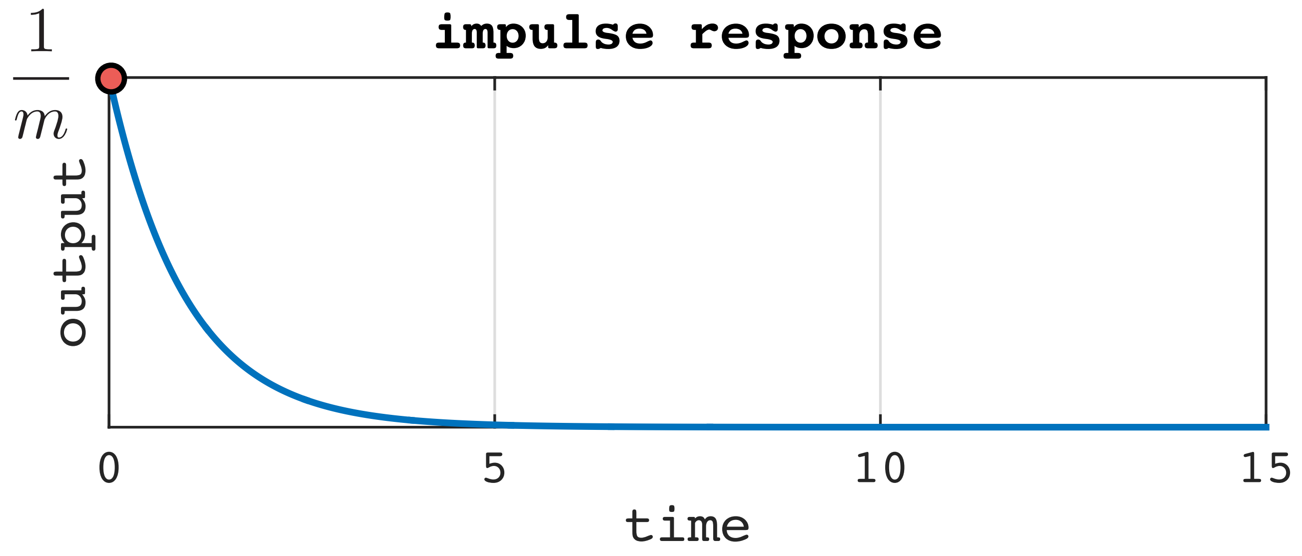
impulse response - example

mass + friction (with coefficient μ), measuring velocity

$$\dot{v} = -\frac{\mu}{m} v + \frac{1}{m} F \quad \xrightarrow[u \,=\, F]{x \,=\, v} \quad \dot{x} = Ax + Bu \qquad \begin{array}{ll} A = -\dfrac{\mu}{m} & B = \dfrac{1}{m} \\ C = 1 & D = 0 \end{array}$$

impulse response $\quad W(t) = Ce^{At}B = \dfrac{1}{m} e^{-\frac{\mu}{m}t}$





physical interpretation (see how much -μ/m is important)

- larger friction coefficient gives faster impulse response exponential decay
- larger mass gives slower impulse response exponential decay

impulse response - example

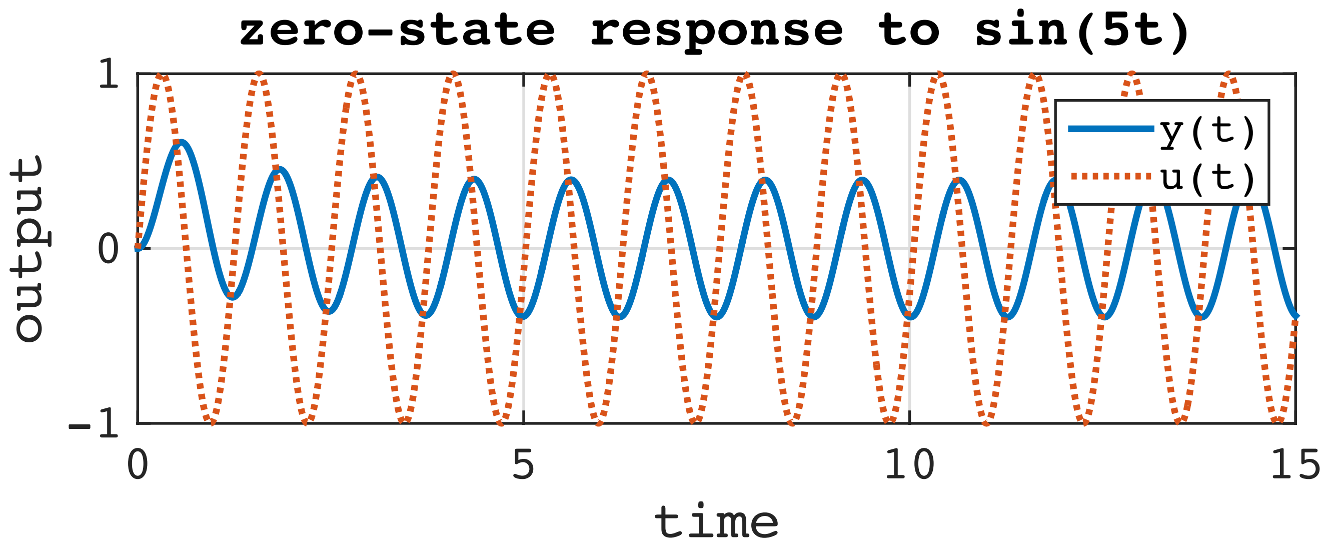
mass + friction (with coefficient μ), measuring velocity

$$\dot{v} = -\frac{\mu}{m} v + \frac{1}{m} F \qquad \text{impulse response} \qquad W(t) = Ce^{At}B = \frac{1}{m} e^{-\frac{\mu}{m} t}$$

for a different input, for example $u(t)$ = $\sin \omega t$, we can compute explicitly the zero-state response from the knowledge of the impulse response

$$y(t) = \int_0^t W(t-\tau)u(\tau)d\tau = \cdots$$

$$= \frac{1}{m} \frac{1}{\left(\frac{\mu}{m}\right)^2 + \omega^2} \left[\frac{\mu}{m} \sin \omega t - \omega \cos \omega t + \omega e^{-\frac{\mu}{m} t}\right]$$



Dirac’s delta: application

similarly for the **state zero-state response**

define $H(t)$ as $H(t) = e^{At}B$

the zero-state response (of the state) can be written as the following convolution integral

$$\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = \int_0^t H(t-\tau)u(\tau)d\tau$$

interpretation

$$\int_0^t H(t-\tau)u(\tau)d\tau \xrightarrow{\text{if } u(t)=\delta(t)} \int_0^t H(t-\tau)\delta(\tau)d\tau$$

$$\int_0^t H(\vartheta)\delta(t-\vartheta)d\vartheta \underset{\uparrow}{=} H(t) \longleftarrow$$

sifting property

$$t-\tau=\vartheta$$
$$\tau=0 \to \vartheta=t$$
$$\tau=t \to \vartheta=0$$

$H(t)$ **state impulse response** $e^{At}B$

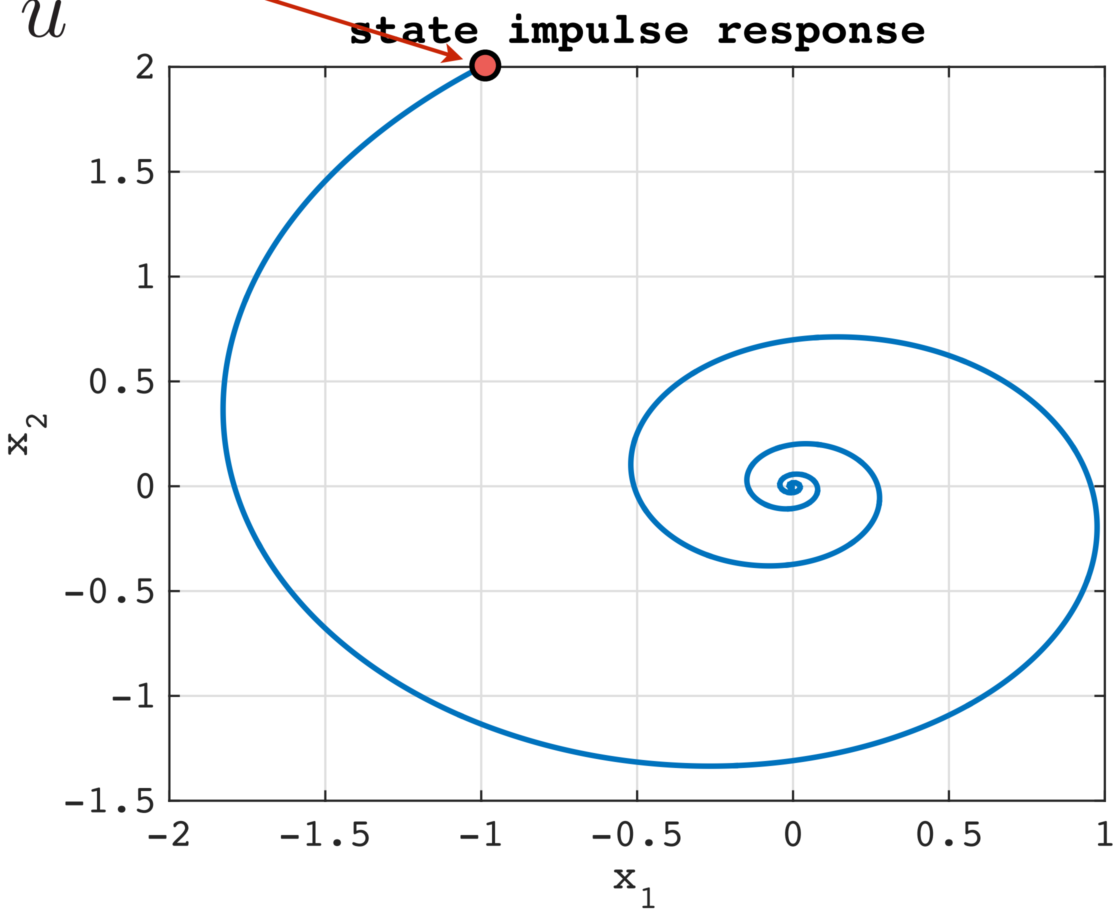
state impulse response

$H(t)$ **state impulse response** $e^{At}B$

other **interpretation** (for SISO systems) $\qquad$ $(=$ ZIR from $x_0)$

$$H(t) = e^{At}B \text{ formally looks like } \boxed{e^{At}x_0} \text{ with } x_0 = B$$

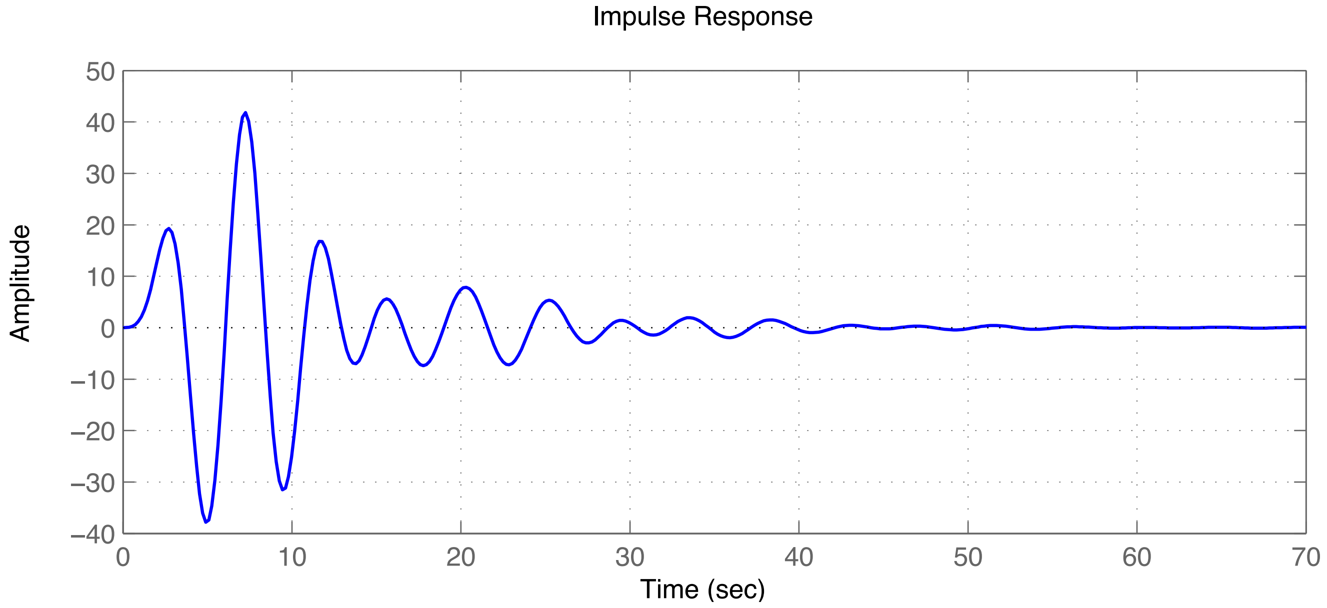
$$\dot{x} = \begin{pmatrix} -0.1 & 0.5 \\ -0.5 & -0.1 \end{pmatrix} x + \boxed{\begin{pmatrix} -1 \\ 2 \end{pmatrix}} u$$



the impulse is transferring the state **instantaneously** from 0 to B, and then evolves with no input as a free evolution from $x_0 = B$

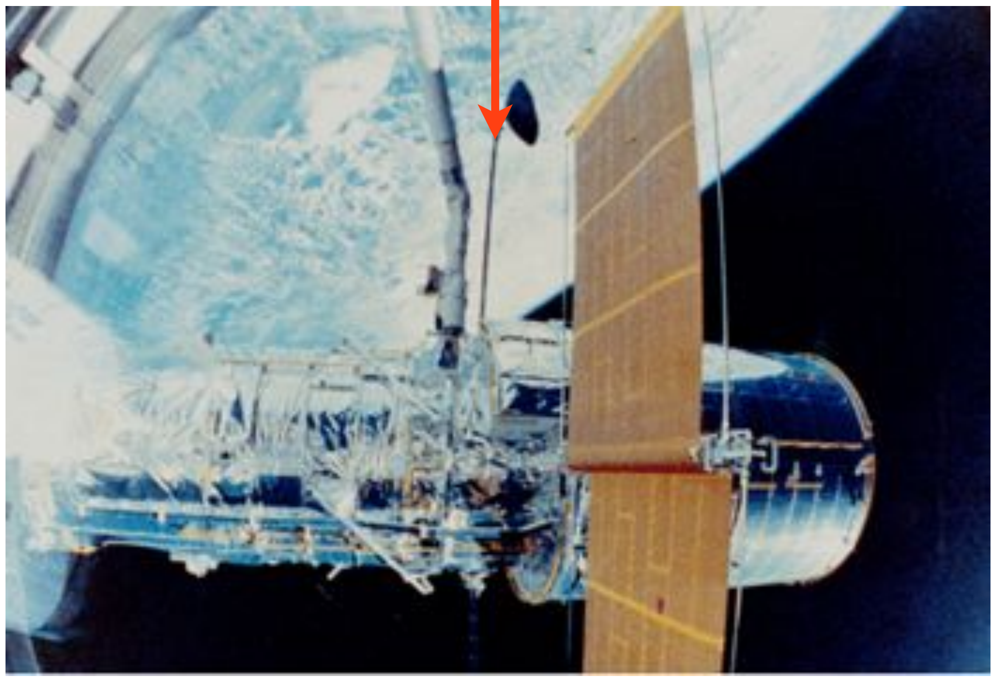
impulse response

impact hammer



impulse response
(here not experimental)

this beam is on the
space telescope here



Hubble Space Telescope

impulse response: experimental determination

Modal testing for vibration analysis:

- the **impulse**, which has an infinitely small duration, is the ideal testing impact to a structure: all vibration modes will be excited with the same amount of energy (more on this in the frequency analysis section)
- the **impact hammer** should be able to replicate this ideal impulse but in reality the strike cannot have an infinitesimal small duration
- the finite duration of the real impact influences the frequency content of the applied force: the longer is the duration the smaller bandwidth

more in the “Mechanical Vibrations” course

solution: total response

more compact notation

$$\boxed{\begin{aligned} x(t) &= \Phi(t)x_0 + \int_0^t H(t-\tau)u(\tau)d\tau \\ y(t) &= \Psi(t)x_0 + \int_0^t W(t-\tau)u(\tau)d\tau \end{aligned}}$$

state	$\Phi(t) = e^{At}$	$H(t) = e^{At}B$
output	$\Psi(t) = Ce^{At}$	$W(t) = Ce^{At}B + D\delta(t)$
	transition matrix	impulse response

Dirac impulse

change of coordinates

in a state space representation

(A, B, C, D) state x

$$\boxed{z = Tx \qquad \det(T) \neq 0}$$

change of coordinates

$(\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D})$ state z

T defines a representation similarity transformation

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \xrightarrow[\text{same system}]{} \begin{aligned} \dot{z} &= \widetilde{A}z + \widetilde{B}u \\ y &= \widetilde{C}z + \widetilde{D}u \end{aligned}$$

input u & output y do not change,
only state is chosen differently

the matrices of the two equivalent system representations are related as

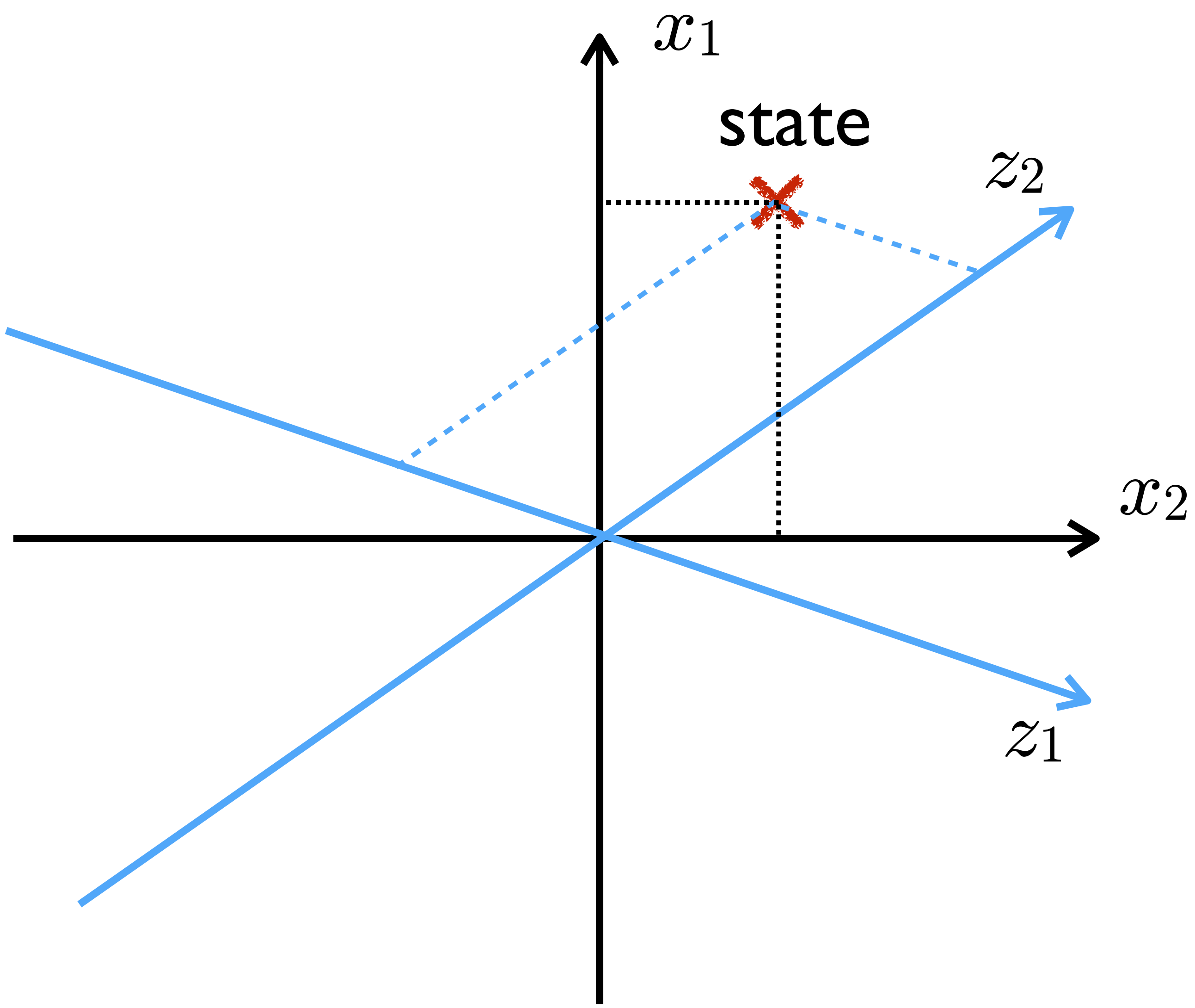
$$\boxed{\widetilde{A} = T\,A\,T^{-1} \qquad \widetilde{B} = T\,B \qquad \widetilde{C} = C\,T^{-1} \qquad \widetilde{D} = D}$$

equivalent system representation (proof) …

change of coordinates

the fact that the same system can be represented with different choices of the state vector is not surprising

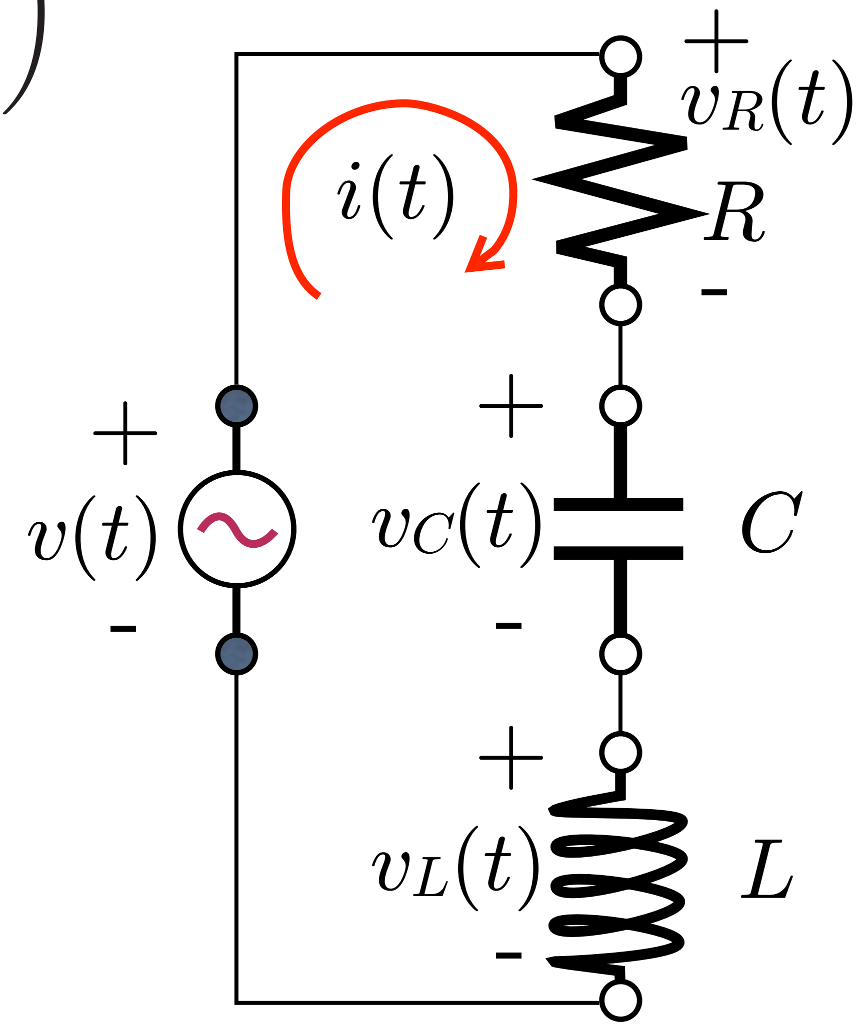
consider the 2-dimensional case, the same state can be represented in the two frames or w.r.t. two different bases



example: from “models of electrical circuits”

series RLC circuit

state $$z(t) = \begin{pmatrix} v_C(t) \\ \dot{v}_C(t) \end{pmatrix} \quad \text{or} \quad x(t) = \begin{pmatrix} i(t) \\ v_C(t) \end{pmatrix}$$



note that $x(t)$ and $z(t)$ are related by a linear nonsingular transformation $z(t) = T\,x(t)$

$$z(t) = \begin{pmatrix} v_C(t) \\ \dot{v}_C(t) \end{pmatrix} = \begin{pmatrix} v_C(t) \\ \frac{1}{C} i(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ \frac{1}{C} & 0 \end{pmatrix}}_{T \text{ nonsingular}} \begin{pmatrix} i(t) \\ v_C(t) \end{pmatrix} = Tx(t)$$

$$T = \begin{pmatrix} 0 & 1 \\ \frac{1}{C} & 0 \end{pmatrix} \longrightarrow T^{-1} = \begin{pmatrix} 0 & C \\ 1 & 0 \end{pmatrix}$$

$$z(t) = T\,x(t) \qquad\qquad x(t) = T^{-1} z(t)$$

we can study the RLC circuit in any equivalent choice of the state vector

models of electrical circuits

series RLC circuit

- with state x we had

$$A = \begin{pmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix}$$

- with state z we have

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ \frac{1}{LC} \end{pmatrix}$$

→ since these two different representations refer to the same RLC circuit, they must share the same important system characteristic

different dynamic matrices but with same characteristics (e.g., same eigenvalues - see algebra slides)

impulse response

1 experiment $\longrightarrow$ 1 response

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

same system
(different representation)

$$\begin{aligned} \dot{z} &= \widetilde{A}z + \widetilde{B}u \\ y &= \widetilde{C}z + \widetilde{D}u \end{aligned}$$

$$W(t) = Ce^{At}B + D\delta(t)$$

$$\widetilde{W}(t) = \widetilde{C}e^{\widetilde{A}t}\widetilde{B} + \widetilde{D}\delta(t)$$

same impulse response

$$\boxed{W(t) = \widetilde{W}(t)}$$

i.e. independent from the chosen set of coordinates (state)

(proof) …

the impulse response characterises the **I/O behavior**

general solution (recap)

$$
\begin{aligned}
x(t) &= \Phi(t)x_0 + \int_0^t H(t-\tau)u(\tau)d\tau \\
y(t) &= \Psi(t)x_0 + \int_0^t W(t-\tau)u(\tau)d\tau
\end{aligned}
$$

$$\Phi(t) = e^{At} \qquad H(t) = e^{At}B \qquad \Psi(t) = Ce^{At} \qquad W(t) = Ce^{At}B + D\delta(t)$$

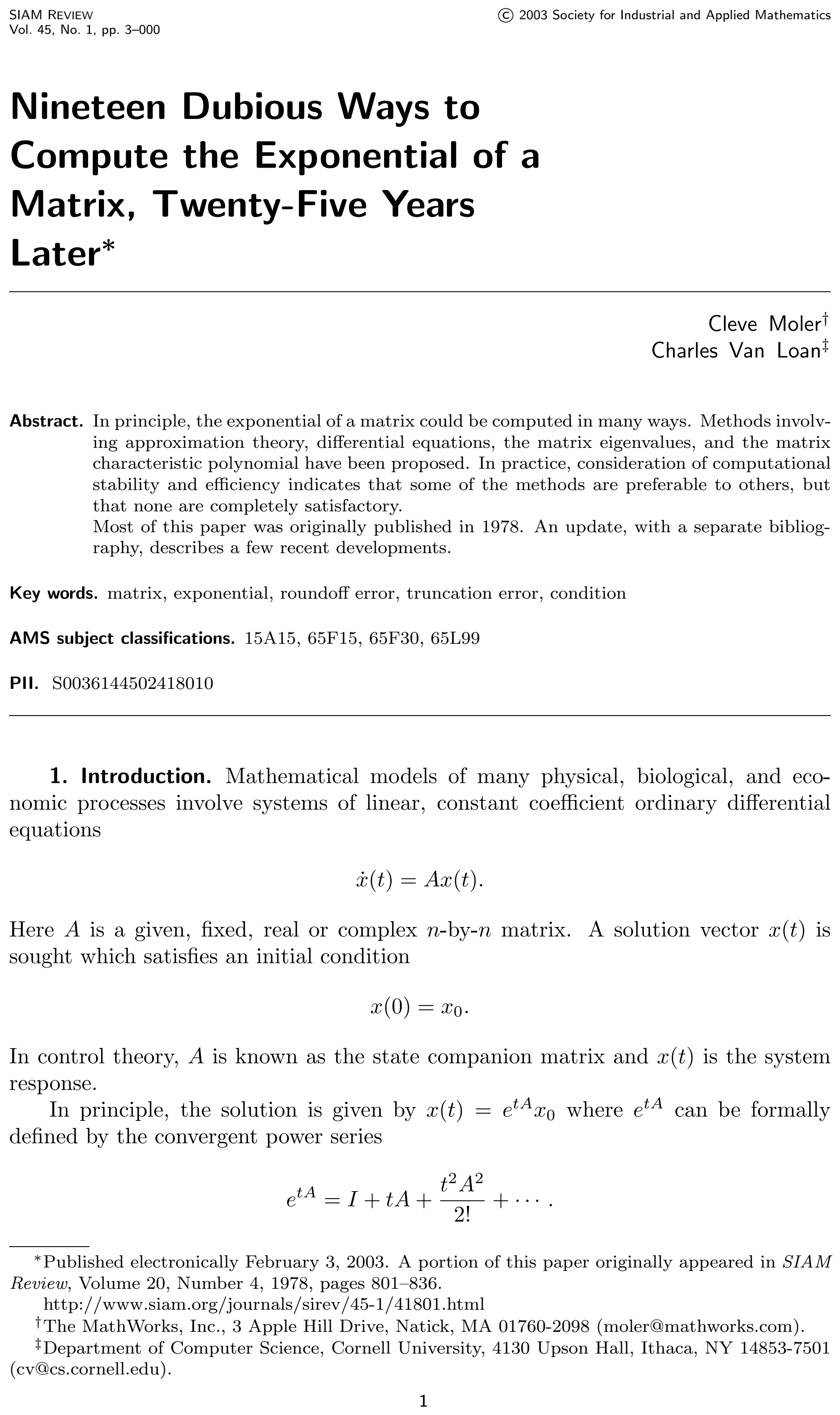
the **matrix exponential** appears everywhere

do we need to compute the exponential using its definition ?

$$e^{At} = I + At + A^2\frac{t^2}{2!} + \cdots = \sum_{k=0}^{\infty} A^k \frac{t^k}{k!}$$

matrix exponential

there are many different ways to compute the matrix exponential ...



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Nineteen Dubious Ways to Compute the Exponential of a Matrix, Twenty-Five Years Later*

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Abstract. In principle, the exponential of a matrix could be computed in many ways. Methods involving approximation theory, differential equations, the matrix eigenvalues, and the matrix characteristic polynomial have been proposed. In practice, consideration of computational stability and efficiency indicates that some of the methods are preferable to others, but that none are completely satisfactory.

Most of this paper was originally published in 1978. An update, with a separate bibliography, describes a few recent developments.

Key words. matrix, exponential, roundoff error, truncation error, condition

AMS subject classifications. 15A15, 65F15, 65F30, 65L99

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1. Introduction. Mathematical models of many physical, biological, and economic processes involve systems of linear, constant coefficient ordinary differential equations

$$\dot{x}(t) = Ax(t).$$

Here A is a given, fixed, real or complex n-by-n matrix. A solution vector $x(t)$ is sought which satisfies an initial condition

$$x(0) = x_0.$$

In control theory, A is known as the state companion matrix and $x(t)$ is the system response.

In principle, the solution is given by $x(t) = e^{tA}x_0$ where e^{tA} can be formally defined by the convergent power series

$$e^{tA} = I + tA + \frac{t^2A^2}{2!} + \cdots .$$

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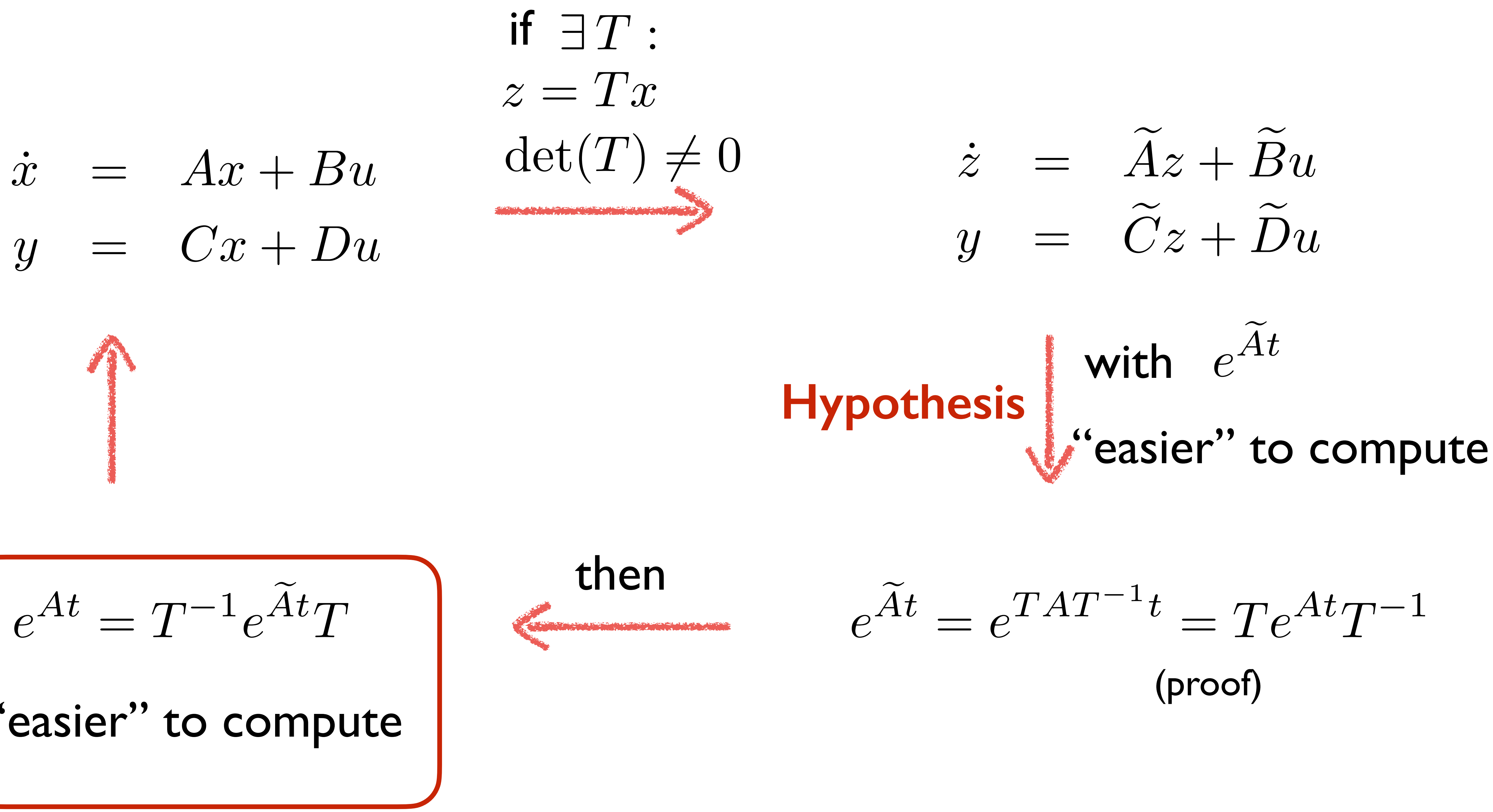
matrix exponential

we will use changes of coordinates (state) $z = T\,x$



if $\exists T$:

$z = Tx$

$\det(T) \neq 0$

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \xrightarrow{\qquad} \begin{aligned} \dot{z} &= \widetilde{A}z + \widetilde{B}u \\ y &= \widetilde{C}z + \widetilde{D}u \end{aligned}$$

Hypothesis

with $e^{\widetilde{A}t}$

"easier" to compute

then

$$e^{\widetilde{A}t} = e^{TAT^{-1}t} = Te^{At}T^{-1}$$

(proof)

$$e^{At} = T^{-1}e^{\widetilde{A}t}T$$

"easier" to compute

since it is used frequently, we prove that

$\boxed{\text{if } \widetilde{A} = TAT^{-1}}$ i.e. $A = T^{-1}\widetilde{A}T$ then $\boxed{e^{At} = T^{-1}e^{\widetilde{A}t}T}$

from $\widetilde{A} = TAT^{-1}$ we obtain $A = T^{-1}\widetilde{A}T$

$$
\begin{aligned}
e^{At} = e^{T^{-1}\widetilde{A}Tt} &= \sum_{k=0}^{\infty} \frac{t^k}{k!}(T^{-1}\widetilde{A}T)^k \\
&= \sum_{k=0}^{\infty} \frac{t^k}{k!} T^{-1}(\widetilde{A})^k T \\
&= T^{-1}\left[\sum_{k=0}^{\infty} \frac{t^k}{k!}(\widetilde{A})^k\right] T \\
&= T^{-1}e^{\widetilde{A}t}T
\end{aligned}
$$

vocabulary

English	Italiano
phase plane	piano delle fasi
ZIR/ZSR	evoluzione libera/forzata
vector field	campo vettoriale
(state) impulse response	risposta impulsiva (nello stato)
convolution integral	integrale di convoluzione
transition matrix	matrice di transizione
sampling property	proprietà di campionamento
superposition principle	principio di sovrapposizione