

Control Systems

Linear Algebra topics

L. Lanari

DIPARTIMENTO DI INGEGNERIA INFORMATICA
AUTOMATICA E GESTIONALE ANTONIO RUBERTI



SAPIENZA
UNIVERSITÀ DI ROMA

outline

- basic facts about matrices
- eigenvalues - eigenvectors - characteristic polynomial - algebraic multiplicity
- eigenvalues invariance under similarity transformation
- invariance of the eigenspace
- geometric multiplicity
- diagonalizable matrix: necessary & sufficient condition
- diagonalizing similarity transformation
- a more convenient similarity transformation for complex eigenvalues
- spectral decomposition
- not diagonalizable (A defective) case: Jordan blocks

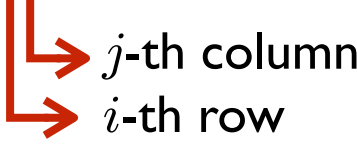
matrices - notations and terminology

$p \times q$ matrix

$$M = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1q} \\ m_{21} & m_{22} & \cdots & m_{2q} \\ \vdots & \vdots & \cdot & \vdots \\ m_{p1} & \cdots & m_{p,q-1} & m_{pq} \end{pmatrix}$$

m_{ij} are the matrix elements (or entries)

$$M = \{m_{ij} : i = 1, \dots, p \quad \& \quad j = 1, \dots, q\}$$



j -th column
 i -th row

transpose

$$M^T = \{m'_{ij} : m'_{ij} = m_{ji}\} \longrightarrow \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

vector (column) $v : n \times 1$

row vector $v^T : 1 \times n$

scalar 1×1

square matrix $M : p \times p$

rectangular matrix $M : p \times q \quad (p \neq q)$

square matrices

diagonal matrix

$$\begin{pmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & m_n \end{pmatrix}$$

compact notation:

$$M = \text{diag}\{m_i\}, \quad i = 1, \dots, p$$

block diagonal matrix

$$\begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{pmatrix}$$

compact notation:

$$M = \text{diag}\{M_i\}, \quad i = 1, \dots, k$$

with M_i square matrix

upper triangular matrix

$$\begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ 0 & m_{22} & \ddots & m_{2n} \\ 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & m_{nn} \end{pmatrix}$$

similarly:

- lower triangular
- strictly lower triangular

upper block triangular matrix

$$\begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix}$$

similarly:

- lower block triangular
- strictly lower block triangular with M_{ii} square matrix

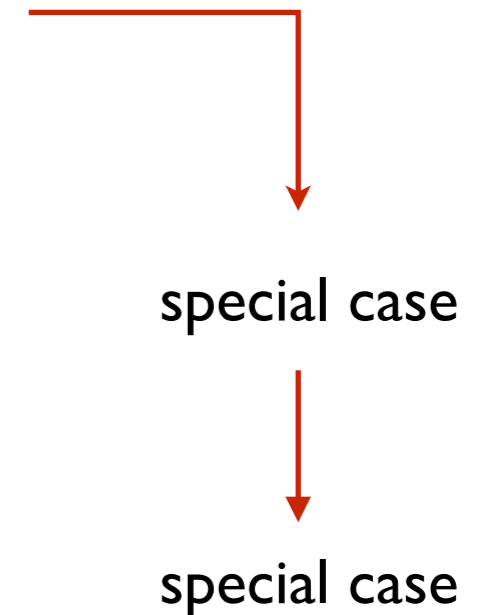
square matrices

- for block triangular or block diagonal the computation of the **determinant** is simplified

$$\det \begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix} = \det(M_{11}) \cdot \det(M_{22})$$

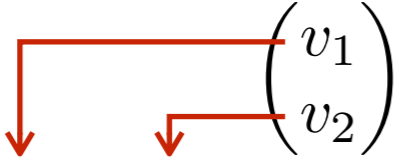
$$\det \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} = \det(M_1) \cdot \det(M_2)$$

- determinant of a diagonal matrix = product of the diagonal terms
- will be useful for the eigenvalue computation



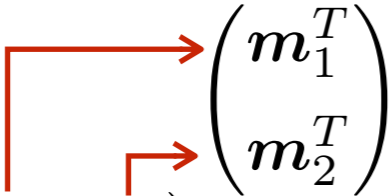
matrix-vector multiplication

- right multiplication of a matrix M by a column vector v is equivalent to making a linear combination of the columns m_i of M with coefficients the elements v_i of v

$$Mv = \begin{pmatrix} m_1 & m_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_1 m_1 + v_2 m_2$$


example $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 6 \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 17 \\ 39 \end{pmatrix}$

- left multiplication of a matrix M by a row vector v^T is equivalent to making a linear combination of the rows m_i^T of M with coefficients the elements v_i of v^T

$$v^T M = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} m_1^T \\ m_2^T \end{pmatrix} = v_1 m_1^T + v_2 m_2^T$$


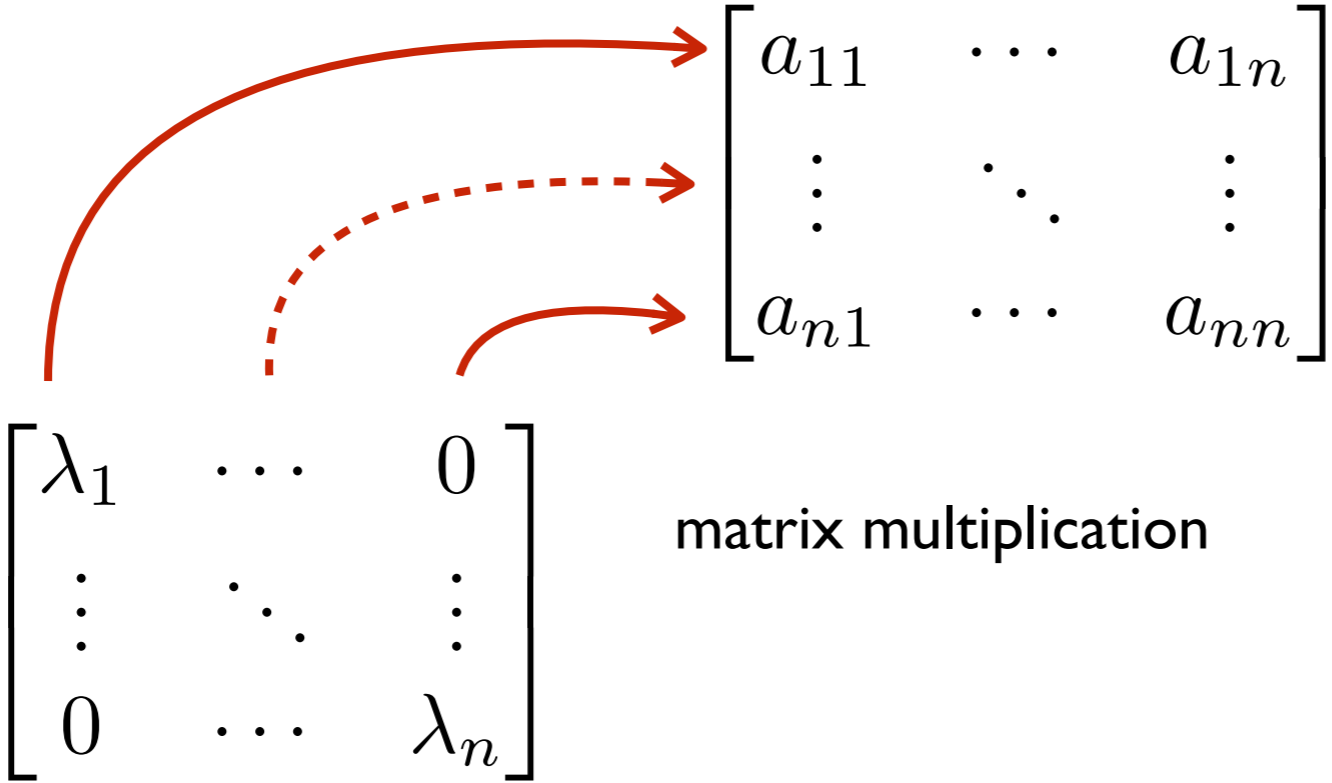
example $\begin{pmatrix} 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 5 \begin{pmatrix} 1 & 2 \end{pmatrix} + 6 \begin{pmatrix} 3 & 4 \end{pmatrix} = \begin{pmatrix} 23 \\ 34 \end{pmatrix}$

square matrices

- multiplication by a diagonal matrix $\Lambda = \text{diag}\{\lambda_i\}$

$$\Lambda A = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}\lambda_1 & \cdots & a_{1n}\lambda_1 \\ \vdots & \ddots & \vdots \\ a_{n1}\lambda_n & \cdots & a_{nn}\lambda_n \end{bmatrix}$$

the i -th element of the diagonal matrix multiplies the i -th row of A



square matrices

- multiplication by a diagonal matrix $\Lambda = \text{diag}\{\lambda_i\}$

$$\Lambda A = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}\lambda_1 & \cdots & a_{1n}\lambda_1 \\ \vdots & \ddots & \vdots \\ a_{n1}\lambda_n & \cdots & a_{nn}\lambda_n \end{bmatrix}$$

the i -th element of the diagonal matrix Λ multiplies the i -th row of A

similarly

$$A \Lambda = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} = \begin{bmatrix} a_{11}\lambda_1 & \cdots & a_{1n}\lambda_n \\ \vdots & \ddots & \vdots \\ a_{n1}\lambda_1 & \cdots & a_{nn}\lambda_n \end{bmatrix}$$

the i -th element of the diagonal matrix Λ multiplies the i -th column of A

square matrices

square matrix **non-singular**
(invertible) $\Leftrightarrow \det(M) \neq 0$

$$(A^{-1})^{-1} = A$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$M = \text{diag}\{m_i\} \implies M^{-1} = \text{diag}\left\{\frac{1}{m_i}\right\}$$

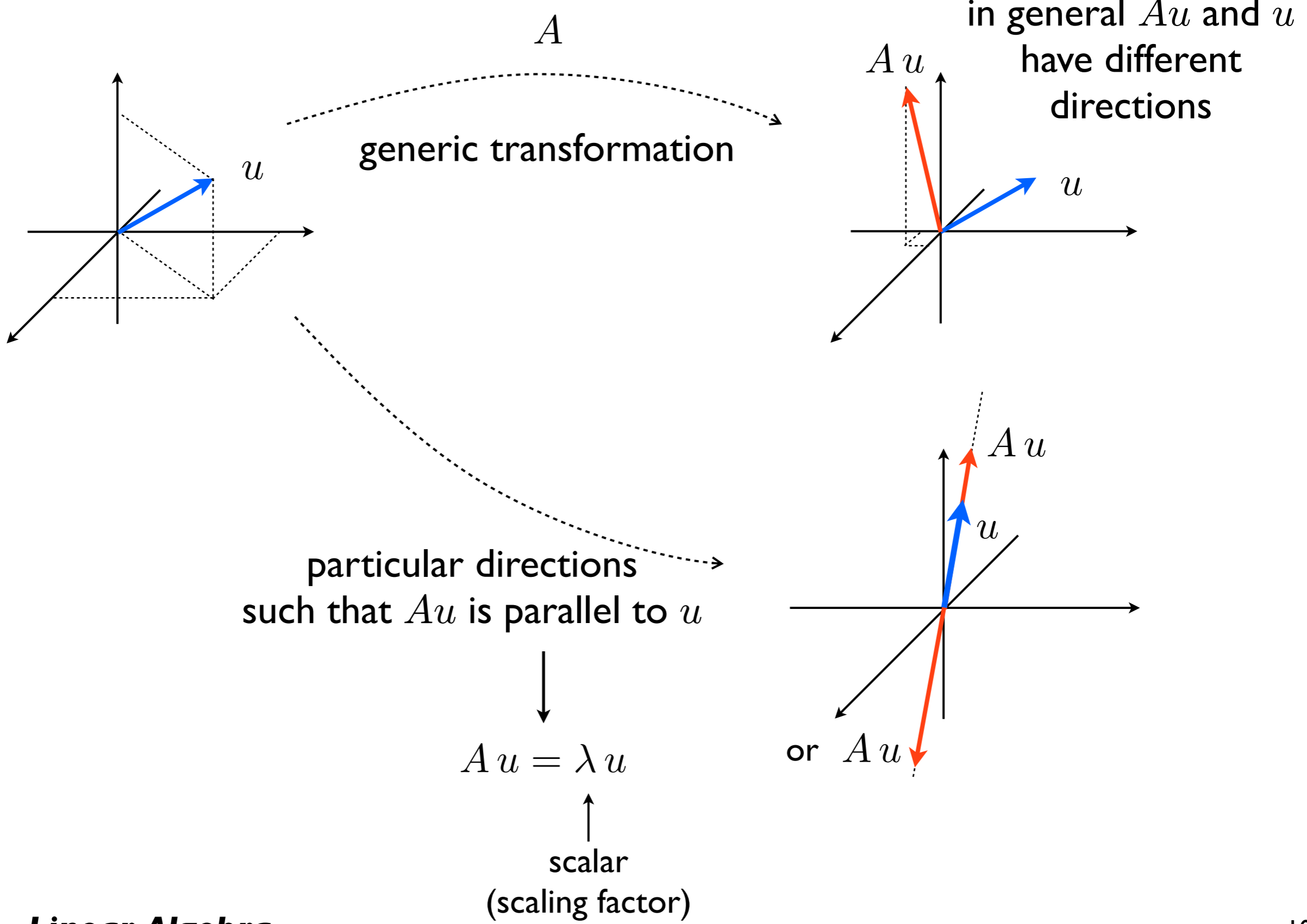
$$A^{-1}A = AA^{-1} = I$$

$$A^0 = I$$

$$\det(A) = \det(A^T) \quad \det(\alpha A) = \alpha^n \det(A) \quad \text{and not } \alpha \det(A)$$

$$\det(MN) = \det(M) \cdot \det(N) \quad \text{for square matrices } M \text{ and } N$$

eigenvalues & eigenvectors



eigenvalues & eigenvectors

example

$$A = \begin{pmatrix} -2 & -2 \\ 2 & 3 \end{pmatrix}$$



eigenvalues

$$\lambda_1 = -1$$

$$\lambda_2 = 2$$



no special relation between the vector and its image

$$A \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

no special relation between the vector and its image

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

the vector and its image are parallel

$$A \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} = -1 \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

scaling factor = eigenvalue

the vector and its image are parallel

$$A \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

eigenvalues & eigenvectors

Definition: given the matrix A ($n \times n$), if for a scalar λ_i there exists a non-zero vector $u_i \neq 0$ such that $A u_i = \lambda_i u_i$ then λ_i is an **eigenvalue** of A and u_i is an associated **eigenvector**

note that $A u_i = \lambda_i u_i \iff (A - \lambda_i I) u_i = 0$ u_i belongs to the nullspace or kernel of $(A - \lambda_i I)$

non-trivial $u_i \neq 0$ solution exists iff $\det(A - \lambda_i I) = 0$

\rightarrow the eigenvalues λ_i are the roots of the n -th order **characteristic polynomial**

$$p_A(\lambda) = \det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \dots + a_1\lambda + a_0$$

the eigenvalues λ_i are the solutions of $p_A(\lambda) = \det(\lambda I - A) = 0$

- same solutions of $\det(A - \lambda I) = 0$ since $\det(A - \lambda I) = (-1)^n \det(\lambda I - A)$
- kernel or **nullspace** of M is the subspace of all vectors v s.t. $Mv = 0$

the eigenvectors u_i associated to the eigenvalue λ_i generate a linear subspace: the **eigenspace** V_i

eigenvalues & eigenvectors

ex. $A = \begin{pmatrix} -2 & -2 \\ 2 & 3 \end{pmatrix}$

$$p_A(\lambda) = \det(\lambda I - A) = \det \begin{pmatrix} \lambda + 2 & 2 \\ -2 & \lambda - 3 \end{pmatrix} = \lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2)$$

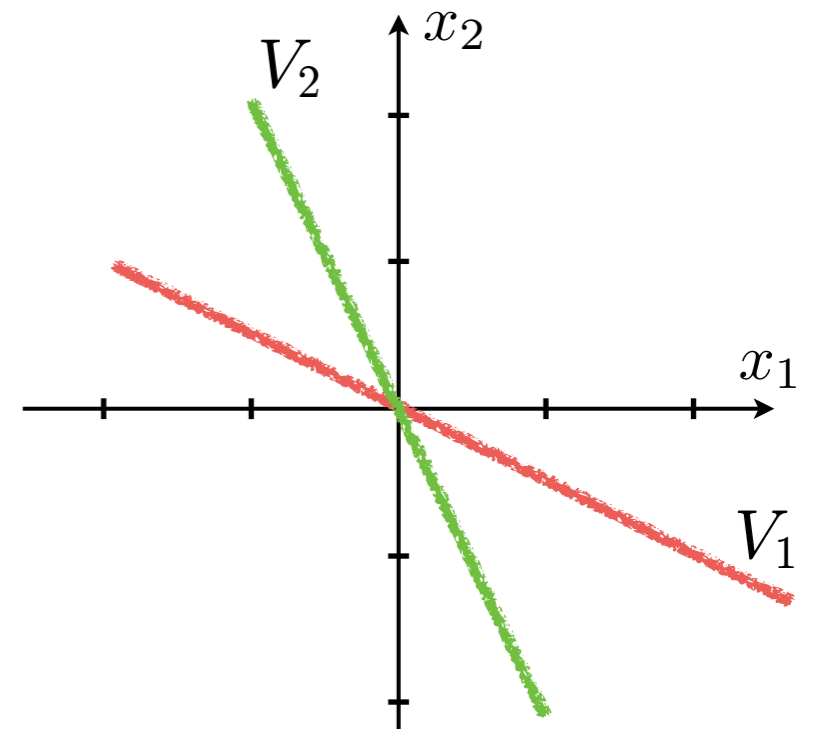
→ therefore the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 2$

- for $\lambda_1 = -1$ the eigenspace V_1 is determined by

$$V_1 = \ker(A - \lambda_1 I) = \ker \begin{pmatrix} -1 & -2 \\ 2 & 4 \end{pmatrix} = \text{gen} \left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}$$

- for $\lambda_2 = 2$ the eigenspace V_2 is determined by

$$V_2 = \ker(A - \lambda_2 I) = \ker \begin{pmatrix} -4 & -2 \\ 2 & 1 \end{pmatrix} = \text{gen} \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$$



eigenvalues & eigenvectors

hyp: A square matrix $n \times n$ with **real** elements

$$p_A(\lambda) = \det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0$$

- $p_A(\lambda)$ is a polynomial of order n with **real** coefficients a_i if A is real
- set of the n solutions of $p_A(\lambda) = 0$ defined as the **spectrum** of A , symbol: $\sigma(A)$
- since the coefficients of $p_A(\lambda)$ are real its roots can be real and/or complex

 generic solution λ_i
(eigenvalue λ_i) $\begin{cases} \lambda_i \in \mathbb{R} \\ \text{or} \\ \lambda_i \in \mathbb{C} \end{cases}$ then also λ_i^* is a solution $\longrightarrow (\lambda_i, \lambda_i^*)$ **pairs**

therefore if $\begin{cases} \lambda_i \in \mathbb{R} \\ \lambda_i \in \mathbb{C} \end{cases}$ then u_i real components
 u_i complex components and $\lambda_i^* \longrightarrow u_i^*$

- define $ma(\lambda_i) =$ **algebraic multiplicity** of eigenvalue λ_i as the multiplicity of the solution $\lambda = \lambda_i$ in $p_A(\lambda) = 0$

eigenvalues & eigenvectors

special cases

**diagonal
matrix**

$$\begin{pmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & m_n \end{pmatrix}$$



eigenvalues = $\{m_i\}$

**triangular
matrix
(upper or lower)**

$$\begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ 0 & m_{22} & \ddots & m_{2n} \\ 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & m_{nn} \end{pmatrix}$$



eigenvalues = $\{m_{ii}\}$

in these situations

eigenvalues = elements on the main diagonal

eigenvalues - invariance

- similarity transformation: $A \xrightarrow[\det(T) \neq 0]{T} TAT^{-1}$ has the same eigenvalues as A

 eigenvalues are **invariant** under similarity transformations (proof)

- right and left eigenvectors:

$$\begin{aligned} Au_i = \lambda_i u_i &\rightarrow u_i && \text{right eigenvector (column)} \\ v_i^T A = v_i^T \lambda_i = \lambda_i v_i^T &\rightarrow v_i^T && \text{left eigenvector (row)} \end{aligned}$$

we will often choose the left eigenvectors such that $v_i^T u_j = \delta_{ij}$

$$\text{where } \delta_{ij} \begin{cases} = 0 & \text{if } i \neq j \\ = 1 & \text{if } i = j \end{cases} \quad \begin{array}{l} \text{Kronecker} \\ \text{delta} \end{array}$$

- $A = \begin{pmatrix} -2 & -2 \\ 2 & 3 \end{pmatrix}$ if $u_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ we will choose $v_1^T = \frac{1}{3} (2 \quad 1)$
- if $u_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ we will choose $v_2^T = \frac{1}{3} (-1 \quad -2)$

eigenspace

the **eigenspace** V_i corresponding to the eigenvalue λ_i of A is the **vector space**

$$V_i = \{u \in \mathbb{R}^n \mid Au = \lambda_i u\} \quad \text{or equivalently} \quad V_i = \text{Ker}(A - \lambda_i I)$$

reminder:

- basis of $\text{Ker}(M)$ is a set of linearly independent vectors which spans the whole subspace $\text{Ker}(M)$
- $\text{span}\{v_1, v_2, \dots, v_k\} =$ vector space generated by all possible linear combinations of the vectors v_1, v_2, \dots, v_k

 as a consequence, the **eigenvector** u_i associated to λ_i is **not unique**

example: $A(\alpha u_i) = \alpha Au_i = \alpha \lambda_i u_i = \lambda_i(\alpha u_i)$

 all belong to the same linear subspace

eigenspace

example:

$$\bullet A = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

$$p_A(\lambda) = \det(\lambda I - A) = (\lambda - 1)^2(\lambda + 1)$$

$$\lambda_1 = 1$$


$$ma(\lambda_1) = 2$$

$$\lambda_2 = -1$$

$$ma(\lambda_2) = 1$$

$$\lambda_1 = 1 \text{ has the eigenspace } V_1 = \ker(A - I) = \ker \begin{pmatrix} -1 & 0 & -1 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{pmatrix} = \text{gen} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

since any linearly independent vectors generated by the chosen basis can be chosen as new basis, V_1 can also be generated by


$$V_1 = \text{gen} \left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ \sqrt{2} \\ 2 \end{pmatrix} \right\}$$

but not as

$$V_1 \neq \text{gen} \left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix} \right\}$$

since, although each vector belongs to V_1 , they are not linearly independent

geometric multiplicity

- if λ_i eigenvalue with $ma(\lambda_i) > 1$ it will have one or more linearly independent eigenvectors u_i

ex. $A_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$, $u_{11} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $u_{12} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ 2 linearly independent eigenvectors
 $\dim(V_1) = 2$

$A_2 = \begin{pmatrix} \lambda_1 & \beta \\ 0 & \lambda_1 \end{pmatrix}$, with $\beta \neq 0$, only $u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ 1 linearly independent eigenvector
 $\dim(V_1) = 1$

note that A_1 and A_2 have the same eigenvalue λ_1 both with same $ma(\lambda_1) = 2$

since they have the same characteristic polynomial $p_A(\lambda) = (\lambda - \lambda_1)^2$

Definition

the **geometric multiplicity** of λ_i is the dimension of the eigenspace associated to λ_i

$$\dim(V_i) = mg(\lambda_i)$$

$$mg(\lambda_i) = \dim [\text{Ker}(A - \lambda_i I)] = n - \text{rank}(A - \lambda_i I)$$

geometric multiplicity

useful
property

$$1 \leq mg(\lambda_i) \leq ma(\lambda_i) \leq n$$



note that

if $ma(\lambda_i) = 1$ then $mg(\lambda_i) = 1$

ex. 1

$$A_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}, \quad (A_1 - \lambda_1 I) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad mg(\lambda_1) = 2 = ma(\lambda_1)$$

$$A_2 = \begin{pmatrix} \lambda_1 & \beta \\ 0 & \lambda_1 \end{pmatrix}, \quad (A_2 - \lambda_1 I) = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \quad mg(\lambda_1) = 1 < ma(\lambda_1)$$

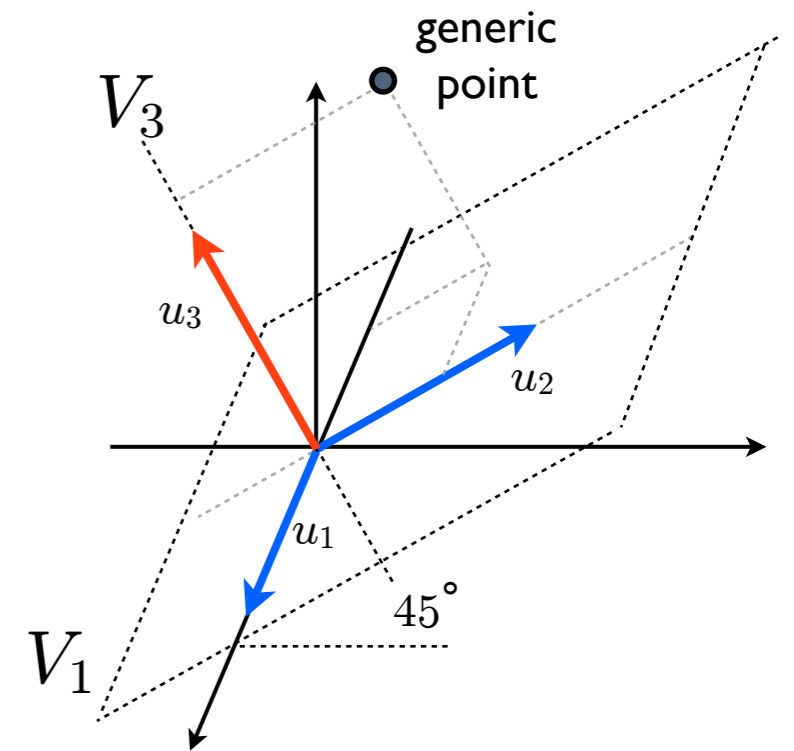
ex. 2

$$P = \begin{pmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \end{pmatrix} \quad \text{projection matrix}$$

$$\lambda_1 = \lambda_2 = 1 \quad u_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad u_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$ma(\lambda_1) = 2 = mg(\lambda_1)$$

$$\lambda_3 = 0 \quad u_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$



diagonalization

Definition

An $(n \times n)$ matrix A is said to be **diagonalizable** if there exists an invertible $(n \times n)$ matrix T such that TAT^{-1} is a diagonal matrix

Theorem

An $(n \times n)$ matrix A is **diagonalizable** if and only if it has n **linearly independent eigenvectors**

since the eigenvalues are invariant under similarity transformations

if A diagonalizable $TAT^{-1} = \Lambda = \text{diag}\{\lambda_i\}, i = 1, \dots, n$


the elements on the diagonal of Λ
are the eigenvalues of A

diagonalization

Hyp: A has n **linearly independent eigenvectors**

(Note: this does not mean necessarily n distinct eigenvalues)

• we need to find T such that TAT^{-1} is a diagonal matrix

since by hypothesis the n eigenvectors u_i are linearly independent, the matrix having the vectors u_i as columns is necessarily non-singular

$$\mathcal{U} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \text{ non-singular } n \times n \text{ matrix}$$

we rewrite the n relations defining the eigenvalues $Au_i = \lambda_i u_i, \quad i = 1, \dots, n$ in matrix form

$$A \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdot & 0 \\ & & \ddots & \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

$$A\mathcal{U} = \mathcal{U}\Lambda$$

with $\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_n\}$

diagonalization

from $A\mathcal{U} = \mathcal{U}\Lambda$ being \mathcal{U} non-singular, we can define T such that $T^{-1} = \mathcal{U}$

$$AT^{-1} = T^{-1}\Lambda \rightarrow A = T^{-1}\Lambda T \rightarrow \Lambda = TAT^{-1}$$

therefore the **diagonalizing similarity transformation** is T s.t.

$$T^{-1} = \mathcal{U} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$$

We have an alternative necessary & sufficient condition for **diagonalizability**

$A: (n \times n)$ is **diagonalizable** if and only if
 $mg(\lambda_i) = ma(\lambda_i)$ for every eigenvalue λ_i

Warning: distinct eigenvalues (real and/or complex) \implies ~~\impliedby~~ A diagonalizable

diagonalization

• $A = \begin{pmatrix} -2 & -2 \\ 2 & 3 \end{pmatrix}$ $T^{-1} = \mathcal{U} = (u_1 \quad u_2) = \begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}$ $T = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix}$

$\lambda_1 = -1$ $\lambda_2 = 2$
 $ma(\lambda_1) = ma(\lambda_2) = 1$ $\frac{1}{3} \begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} -2 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$

note that: since the algebraic multiplicity is 1 necessarily also the geometric one is 1

• $A = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix}$ $V_1 = \text{gen} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ $V_2 = \text{gen} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

$\lambda_1 = 1$ $ma(\lambda_1) = 2 = mg(\lambda_1)$
 $\lambda_2 = -1$ $ma(\lambda_2) = 1 = mg(\lambda_2)$

$T^{-1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$ $T = \begin{pmatrix} -1/2 & 1 & -1/2 \\ 1/2 & 0 & -1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix} \Rightarrow \Lambda = TAT^{-1} = \mathcal{U}^{-1}A\mathcal{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

diagonalization

As already noted, A having **distinct eigenvalues** implies that A is **diagonalizable** (**sufficient condition**) but the A diagonalizable does not necessarily imply that its eigenvalues are distinct

Sufficiency comes from the fact that distinct eigenvalues (real and/or complex) generate linearly independent eigenvectors

However the condition is **not necessary**, see for example

- $A = \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i \end{pmatrix}$ diagonal but coincident eigenvalues, $ma(\lambda_i) = 2$

however there are 2 linearly independent eigenvectors for the eigenvalue λ_i and hence the geometric multiplicity is also 2

- $A = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix}$ diagonalizable but not distinct eigenvalues

similarly the eigenvalue $\lambda_1 = 2$ with algebraic multiplicity 2 has also $mg(\lambda_1) = 2$

diagonalization: complex eigenvalues case

hyp: A square matrix $n \times n$ with **real** elements

We know complex eigenvalues for A real come necessarily in pairs (λ_i, λ_i^*)
 the complex eigenvalue λ_i will have eigenvector u_i with complex elements

$$\begin{array}{l} \text{consider the case } n = 2 \quad \lambda_i = \alpha_i + j\omega_i \quad \text{with its eigenvector} \quad u_i = u_{ai} + ju_{bi} \\ \lambda_i^* = \alpha_i - j\omega_i \quad \longrightarrow \quad u_i^* = u_{ai} - ju_{bi} \end{array}$$

2 choices

- **diagonalization** (since the eigenvalues are distinct, A is diagonalizable)

$$T^{-1} = [u_i \quad u_i^*] \quad \rightarrow \quad D_i = TAT^{-1} = \begin{bmatrix} \lambda_i & 0 \\ 0 & \lambda_i^* \end{bmatrix} \quad \text{but complex elements}$$

- or **real block** 2×2 (no diagonalization)

$$T_R^{-1} = [u_{ai} \quad u_{bi}] \quad \rightarrow \quad M_i = T_R A T_R^{-1} = \begin{bmatrix} \alpha_i & \omega_i \\ -\omega_i & \alpha_i \end{bmatrix} \quad \text{real elements}$$

linearly independent when A is real

real system representation for complex eigenvalues

diagonalization: complex eigenvalues case

- $A = \begin{pmatrix} 0 & -2 \\ 1 & 2 \end{pmatrix}$

$$p_A(\lambda) = \lambda^2 - 2\lambda + 2 = (\lambda - 1 - j)(\lambda - 1 + j)$$

$$\lambda_1 = 1 + j \quad u_1 = \begin{pmatrix} 2 \\ -1 - j \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + j \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\lambda_1^* = 1 - j \rightarrow u_1^*$$

therefore $T_R^{-1} = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}$ $T_R = \begin{bmatrix} 1/2 & 0 \\ -1/2 & -1 \end{bmatrix}$ and $T_R A T_R^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & \omega \\ -\omega & \alpha \end{bmatrix}$

- $A = \begin{pmatrix} 0 & 1 & | & 0 & 0 \\ -2 & 2 & | & 0 & 0 \\ \hline 1 & 1 & | & -2 & 1 \\ 2 & 1 & | & -2 & 0 \end{pmatrix}$

$$\lambda_1 = 1 + j \rightarrow u_1 = \begin{pmatrix} -1 + j \\ -2 \\ -1 + j \\ -1 + j \end{pmatrix}$$

$$ma(\lambda_1) = 1 = mg(\lambda_1)$$

$$\lambda_2 = -1 + j \rightarrow u_2 = \begin{pmatrix} 0 \\ 0 \\ 1 - j \\ 2 \end{pmatrix}$$

$$ma(\lambda_2) = 1 = mg(\lambda_2)$$

$$T_R^{-1} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 2 & 0 \end{pmatrix}$$

$$T_R = \begin{pmatrix} 0 & -1/2 & 0 & 0 \\ 1 & -1/2 & 0 & 0 \\ -1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & -1 & 1/2 \end{pmatrix}$$

$$T_R A T_R^{-1} = \begin{pmatrix} 1 & 1 & | & 0 & 0 \\ -1 & 1 & | & 0 & 0 \\ \hline 0 & 0 & | & -1 & 1 \\ 0 & 0 & | & -1 & -1 \end{pmatrix}$$

check with Matlab

diagonalization

- simultaneous presence of real and complex eigenvalues

If A **diagonalizable**, there exists a non-singular matrix R such that

$$RAR^{-1} = \text{diag} \{ \Lambda_r, M_{r+1}, M_{r+3}, \dots, M_{q-1} \}$$

with $\Lambda_r = \text{diag} \{ \lambda_1, \dots, \lambda_r \}$ for real eigenvalues

$$M_i = \begin{bmatrix} \alpha_i & \omega_i \\ -\omega_i & \alpha_i \end{bmatrix} \quad \text{for each complex pair of eigenvalues } (\lambda_i, \lambda_i^*)$$

and $T_R^{-1} = (u_1 \quad \dots \quad u_r \quad \text{Re}[u_{r+1}] \quad \text{Im}[u_{r+1}] \quad \dots \quad \text{Re}[u_{q-1}] \quad \text{Im}[u_{q-1}])$

- $A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -2 & 1 \\ 1 & -1 & -1 \end{pmatrix}$

$$\lambda_{1,2} = -1 \pm j$$

$$\lambda_3 = -1$$

Matlab code

```
A = [0,0,-1;0,-2,1;1,-1,-1];
[V,D] = eig(A)
Tinv = [real(V(:,1)),imag(V(:,1)),V(:,3)];
inv(Tinv)*A*Tinv
```

$$\rightarrow \begin{pmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

diagonalization

example

$$A = \begin{bmatrix} -6 & 5 & 4 \\ 5 & -6 & -16 \\ -10 & 10 & 9 \end{bmatrix} \quad \lambda_1 = -1$$
$$\lambda_{2/3} = -1 \pm 10j$$

$$\lambda_1 = -1 \quad u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = -1 + 10j \quad u_2 = \begin{bmatrix} 0.5 + 0.1j \\ -0.5 + 1.1j \\ 1 \end{bmatrix} \quad u_{2a} = \begin{bmatrix} 0.5 \\ -0.5 \\ 1 \end{bmatrix} \quad u_{2b} = \begin{bmatrix} 0.1 \\ 1.1 \\ 0 \end{bmatrix}$$

$$T^{-1} = [u_1 \quad u_{2a} \quad u_{2b}] = \begin{bmatrix} 1 & 0.5 & 0.1 \\ 1 & -0.5 & 1.1 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow TAT^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 10 \\ 0 & -10 & -1 \end{bmatrix}$$

A diagonalizable: spectral decomposition

Hyp: A diagonalizable

$$A = U \Lambda U^{-1}$$

columns (linearly independent)

$$U = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \longrightarrow U^{-1} = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix}$$

rows

these are also left eigenvectors

$$U^{-1}U = I \implies v_i^T u_j = \delta_{ij}, \quad i, j = 1, \dots, n$$

here, the choice of the left eigenvector should be so that this condition is met

therefore computing explicitly

$$A = U \Lambda U^{-1} = \begin{bmatrix} \lambda_1 u_1 & \lambda_2 u_2 & \cdots & \lambda_n u_n \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix}$$

spectral form of A

$$A = \sum_{i=1}^n \lambda_i u_i v_i^T$$

or **eigen-decomposition**

A diagonalizable: spectral decomposition

$$A = \sum_{i=1}^n \lambda_i \underbrace{u_i}_{\text{column}} \underbrace{v_i^T}_{\text{row}} \quad \left. \vphantom{\sum} \right\} (n \times n) \longrightarrow P_i = u_i v_i^T$$

is the **projection matrix** on the invariant subspace generated by u_i

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\lambda_1 = 2$$

$$\lambda_2 = 1$$

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$v_1^T = (1 \quad 1)$$

$$P_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

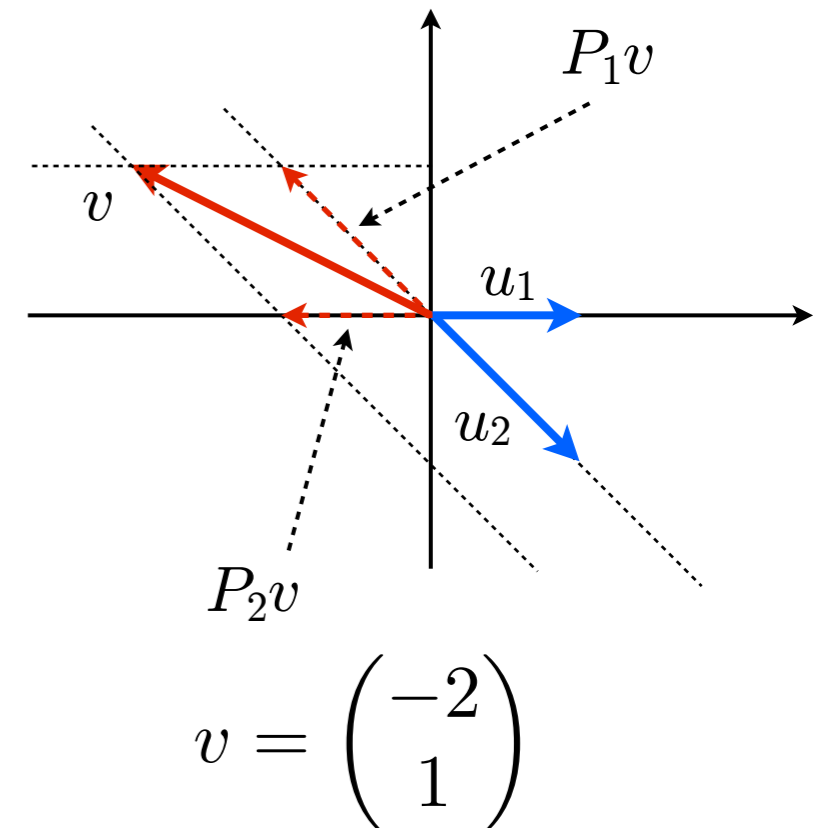
$$P_1 v = -1 \cdot u_1$$

$$u_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$v_2^T = (0 \quad -1)$$

$$P_2 = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$$

$$P_2 v = 1 \cdot u_2$$



non-diagonalizable case

for illustrative purposes, we assume that λ_i is the only eigenvalue and therefore the algebraic multiplicity will be equal to n if A is $n \times n$

since A is not diagonalizable we have $0 < mg(\lambda_i) < ma(\lambda_i) = n$

in this case A is said to be **defective**

→ then there exists a change of coordinates such that in $T A T^{-1}$ the eigenvalue λ_i will have associated $mg(\lambda_i)$ Jordan blocks J_k with $k = 1, \dots, mg(\lambda_i)$, each of dimension n_k (we will not explore how to determine n_k)

single
Jordan block
of dimension n_k

$$J_k = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_i & 1 \\ 0 & \cdots & \cdots & 0 & \lambda_i \end{bmatrix} \in \mathbb{R}^{n_k \times n_k}$$

- note that the null space of $J_k - \lambda_i I$ has dimension 1
- the dimension of the largest Jordan block associated to λ_i is called **index**

non-diagonalizable case (A defective)

In this case the eigenvector is replaced by the generalized eigenvector which will form chains of n_k generalized eigenvectors (not part of this course)

The resulting matrix, after a proper change of coordinates, is the **Jordan canonical form** which is block diagonal matrix having the Jordan blocks on the main diagonal

- example:

unique eigenvalue λ_i of matrix A ($n \times n$) with geometric multiplicity $mg(\lambda_i) = p$

and algebraic multiplicity $ma(\lambda_i) = n = \sum_{k=1}^p n_k$

then there exists a nonsingular matrix T such that $TAT^{-1} = J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_p \end{bmatrix}$

with $J_k \in \mathbb{R}^{n_k \times n_k}$ Jordan block of dim n_k for each $k = 1, \dots, p$

non-diagonalizable case (multiple c.c. eigenvalues)

What happens if we have repeated complex conjugate eigenvalues in the non-diagonalizable case, i.e., $mg(\lambda_i) < ma(\lambda_i)$?

Let's consider, for illustrative purposes, $n = 4$ with the repeated pair of complex conjugate eigenvalues (λ_i, λ_i^*) , $\lambda_i = \alpha_i + j\omega_i$ with $ma(\lambda_i) = 2$ and $mg(\lambda_i) = 1$

→ we know there exists a change of coordinates T which will lead to one Jordan block of dimension 2 for each eigenvalue

$$TAT^{-1} = \begin{pmatrix} \lambda_i & 1 & \vdots & 0 & 0 \\ 0 & \lambda_i & \vdots & 0 & 0 \\ \hline 0 & 0 & \vdots & \lambda_i^* & 1 \\ 0 & 0 & \vdots & 0 & \lambda_i^* \end{pmatrix}$$

block diagonal complex

→ however we have the usual problem that the resulting matrix is complex; nevertheless as in the diagonalizable case, there exists a change of coordinates T_R which leads to a real matrix

$$T_RAT_R^{-1} = \begin{pmatrix} \alpha_i & \omega_i & \vdots & 1 & 0 \\ -\omega_i & \alpha_i & \vdots & 0 & 1 \\ \hline 0 & 0 & \vdots & \alpha_i & \omega_i \\ 0 & 0 & \vdots & -\omega_i & \alpha_i \end{pmatrix}$$

block triangular real

special cases

- if $ma(\lambda_i) = 1$ then $mg(\lambda_i) = 1$
- if $mg(\lambda_i) = 1$ then only one Jordan block of dimension $ma(\lambda_i)$

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \quad \lambda_1 = -1 \quad ma(\lambda_1) = 2 \quad mg(\lambda_1) = 1$$

- if λ_i unique eigenvalue of matrix A and if $\text{rank}(\lambda_i I - A) = ma(\lambda_i) - 1$ then only one Jordan block of dimension $ma(\lambda_i)$

consequence of the **rank-nullity theorem** applied to $A - \lambda_i I$

$$A - \lambda_i I : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$\dim(\mathbb{R}^n) = \dim(\text{Ker}(A - \lambda_i I)) + \dim(\text{Im}(A - \lambda_i I))$$

n

nullity($A - \lambda_i I$)

rank($A - \lambda_i I$)

summary

<p>A real diagonalizable</p> <p>$mg(\lambda_i) = ma(\lambda_i)$ for all i</p>	<p>$\exists T$ s.t. $TAT^{-1} = \Lambda$</p> <p>$\Lambda = \text{diag}\{\lambda_i\}$</p> <hr/> <p>$A = \sum_{i=1}^n \lambda_i u_i v_i^T$</p>	<p>for real & complex λ_i</p> <p>$\Lambda_r = \text{diag}\{\lambda_1, \dots, \lambda_r\}$</p> <p>alternative choice for complex (λ_i, λ_i^*)</p> <p>$M_i = \begin{bmatrix} \alpha_i & \omega_i \\ -\omega_i & \alpha_i \end{bmatrix}$</p> <hr/> <p>spectral form</p>
<p>A real not diagonalizable</p> <p>$mg(\lambda_i) < ma(\lambda_i)$</p>	<p>$\exists T$ s.t.</p> <p>$TAT^{-1} = \text{diag}\{J_k\}$</p> <p>block diagonal</p>	<p>$mg(\lambda_i)$ Jordan blocks of the form</p> <p>$J_k = \begin{bmatrix} \lambda_i & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \lambda_i & 1 \\ 0 & \dots & 0 & \lambda_i \end{bmatrix} \in \mathbb{R}^{n_k \times n_k}$</p>

vocabulary

English	Italiano
eigenvalue/eigenvector	autovalore/autovettore
characteristic polynomial	polinomio caratteristico
algebraic/geometric multiplicity	molteplicità algebraica/geometrica
similar matrix	matrice simile
spectral form	forma spettrale