Control Systems

Internal Stability - LTI systems

L. Lanari

Dipartimento di Ingegneria Informatica
Automatica e Gestionale Antonio Ruberti

Sapienza
Università di Roma

outline

LTI systems:

- definitions
- conditions
- Routh stability criterion
- equilibrium points

Nonlinear systems:

- equilibrium points
- examples
- stable equilibrium state (see slides StabilityTheory by Prof. G. Oriolo)
- indirect method of Lyapunov (see slides StabilityTheory by Prof. G. Oriolo)

linear systems - equilibrium states

the **origin** is a particular state:

- at the origin the state velocity is 0 if no inputs are applied
- therefore if we start from the origin, the state will stay there in the ZIR
- mathematically $0 = A.0$

we can look for any state x_e with such a property i.e. a state x_e such that

$$Ax_e = 0$$

these are defined as **equilibrium states**

all the equilibrium states of a LTI system belong to the nullspace of A

- if A nonsingular then only one equilibrium state (the origin)
- if A singular then infinite equilibrium states (subspace)

note that A singular means

$$\det(A) = \det(A - 0.I) = 0$$

that is $\lambda_i = 0$ is an eigenvalue of A

linear systems - equilibrium states

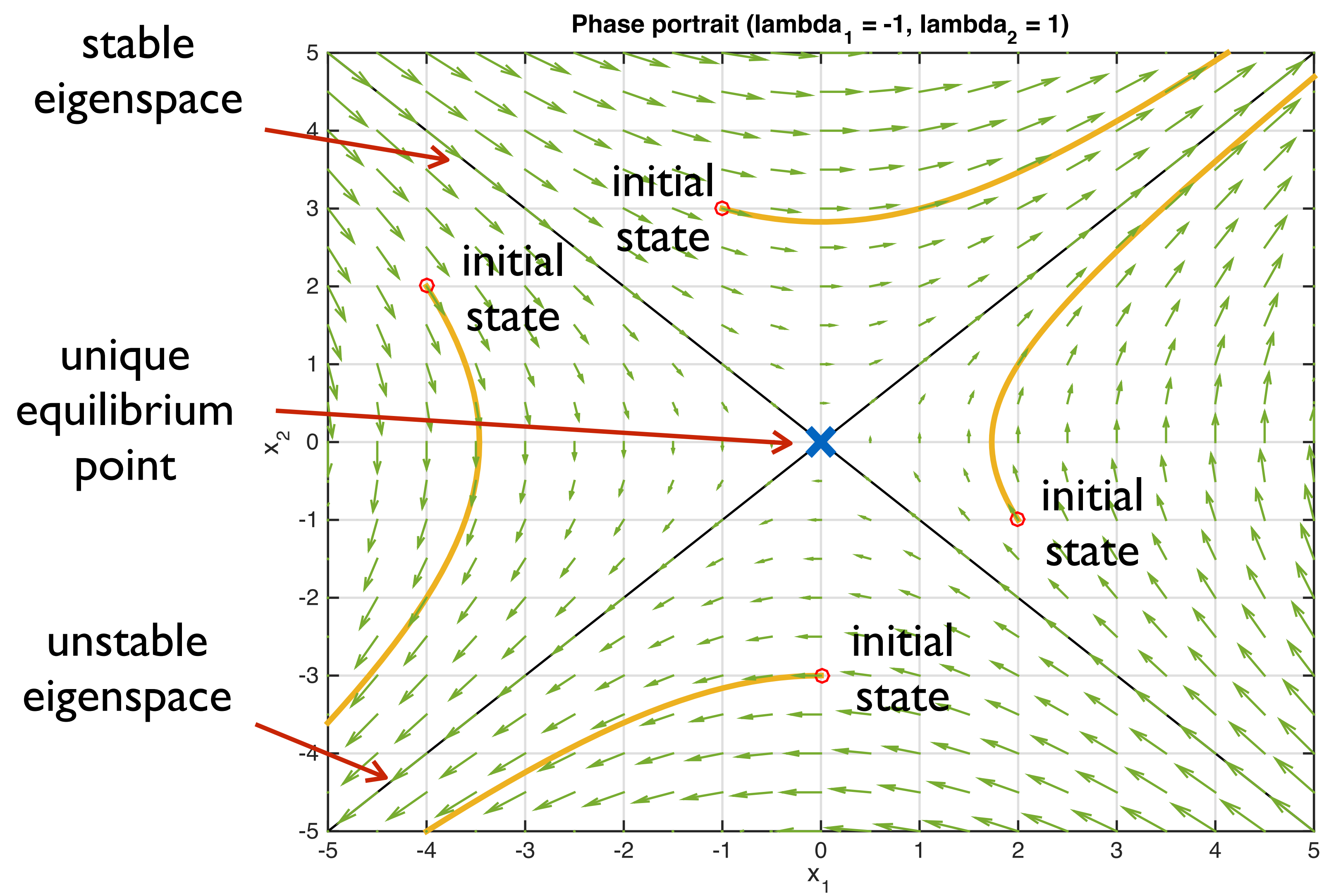
therefore

- if A has no eigenvalue $\lambda_i = 0$ then the system has a **unique equilibrium point** which is necessarily the **origin** (physical example: MSD system)
- if A has at least one eigenvalue $\lambda_i = 0$ then the system has **infinite equilibrium points** (physical example: point mass with friction)

example 1

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\lambda_1 = -1 \quad \lambda_2 = 1$$



linear systems - equilibrium states

example 2

$$A = \begin{pmatrix} -0.25 & 0.25 \\ 0.25 & -0.25 \end{pmatrix}$$

every equilibrium state is of the form $x_e = \begin{pmatrix} x_{1e} \\ x_{2e} = x_{1e} \end{pmatrix}$

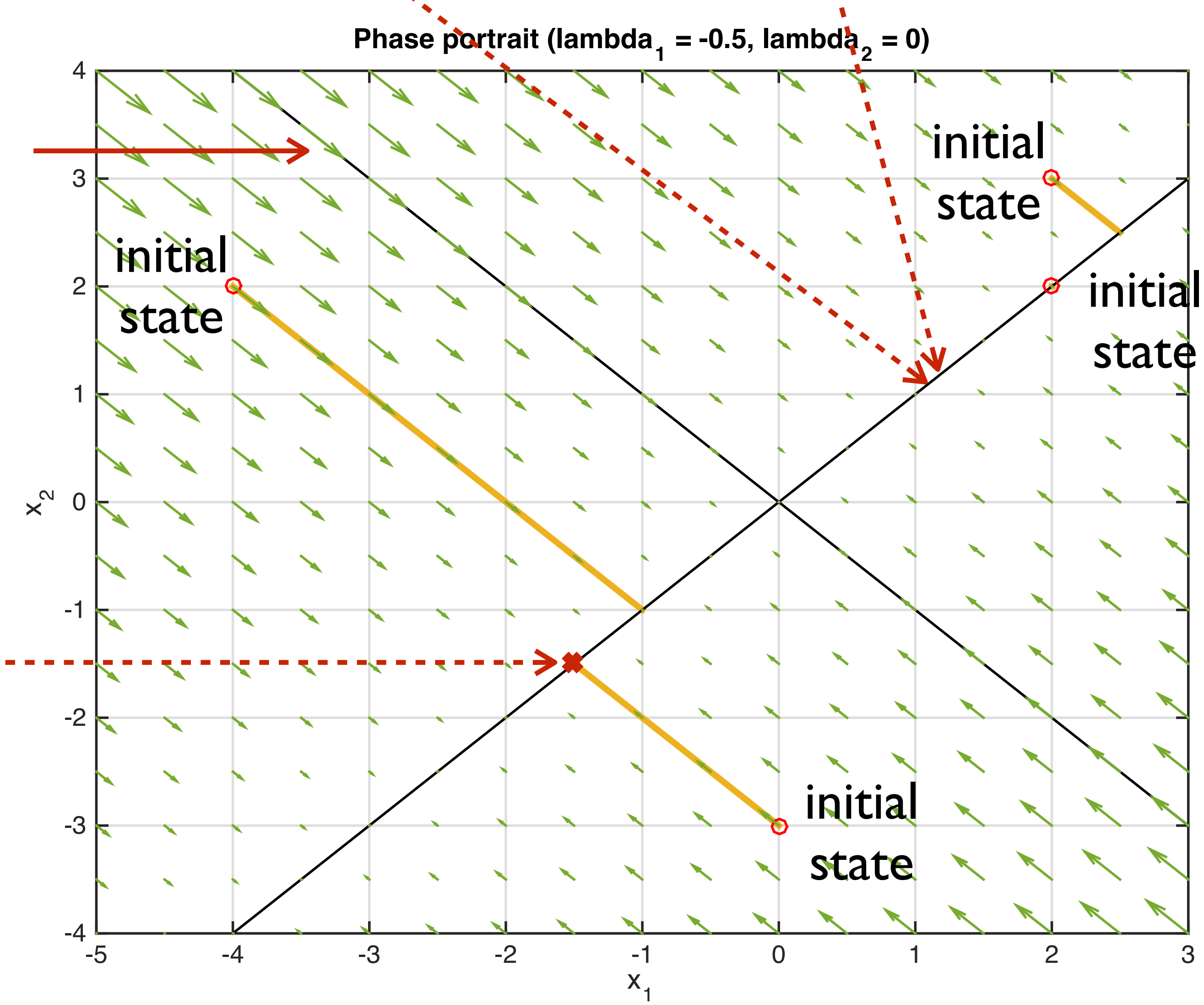
eigenspace relative to $\lambda_i = 0$ = **infinite** equilibrium points

$$\det(A) = 0$$

$$\lambda_1 = -0.5 \quad \lambda_2 = 0$$

$$\lambda_2 = 0 \longrightarrow u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

stable eigenspace



the ZIR for arbitrary initial conditions will not always tend to the origin: following the velocity directions, we end in an equilibrium point (✖) different from the origin

definitions (LTI systems)

(AS) - A system S is said to be **asymptotically stable** if its state zero-input response **converges** to the origin for **any** initial condition

(MS) - A system S is said to be **(marginally) stable** if its state zero-input response remains **bounded** for **any** initial condition

(U) - A system S is said to be **unstable** if its state zero-input response **diverges** for **some** initial condition

note: only interested in the free state evolution (ZIR)

note: use of "any/some"

$\longrightarrow$ state transition matrix $\Phi(t) = e^{At}$ LTI (Linear Time Invariant)

$\Phi(t, t_0)$ LTV (Linear Time Variant)

possible behaviors

we saw that the $x_{ZIR}(t) = e^{At}x_0$ is a linear combination of

A diagonalizable $mg(\lambda_i) = ma(\lambda_i)$ for all i	real λ_i	**aperiodic modes** $e^{\lambda_i t}$
	complex $\lambda_i = \alpha_i + j\omega_i$	**pseudoperiodic modes** $e^{\alpha_i t}\left[\sin(\omega_i t + \varphi_R)u_{\mathrm{re}} + \cos(\omega_i t + \varphi_R)u_{\mathrm{im}}\right]$
A not diagonalizable (defective matrix A) $mg(\lambda_i) < ma(\lambda_i)$	real λ_i	$\ldots, \dfrac{t^{n_k-1}}{(n_k-1)!}e^{\lambda_i t}$
	complex $\lambda_i = \alpha_i + j\omega_i$	$\ldots, \dfrac{t^{n_k-1}}{(n_k-1)!}e^{\alpha_i t}\sin\omega_i t$

max dimension of Jordan block J_k

stability and eigenvalues (stability criterion)

A LTI system is **asymptotically stable**
if and only if
all the eigenvalues have **strictly negative real part**

A LTI system is **(marginally) stable**
if and only if
all the eigenvalues have **non positive** real part
and those which have **zero real part** have **scalar Jordan blocks**

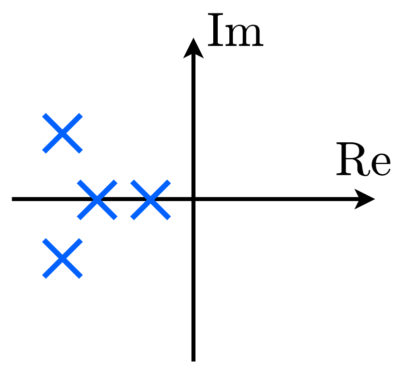
equivalent to $mg(\lambda_i) = ma(\lambda_i)$ for all λ_i with 0 real part

A LTI system is **unstable**
if and only if
there exists at least one eigenvalue with **positive real part** or a
Jordan block corresponding to an eigenvalue with **zero real part** of dimension
greater than 1

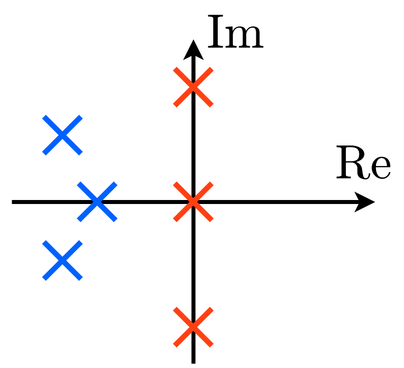
equivalent to $mg(\lambda_i) < ma(\lambda_i)$ for all λ_i with 0 real part

stability and eigenvalues (stability criterion)

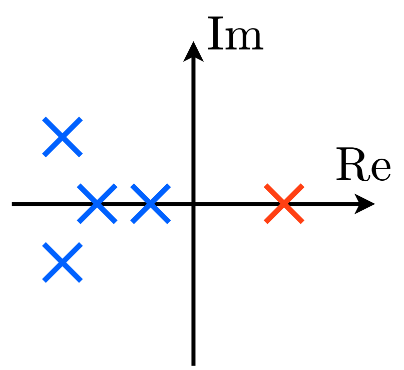
it all depends upon the positioning of the eigenvalues of matrix A in the complex plane



asymptotic stability
all eigenvalues in the open left half-plane



distinct eigenvalues case
(marginal) stability
some eigenvalues may be on the Im axis
$(ma(\lambda_i) = 1$ case)

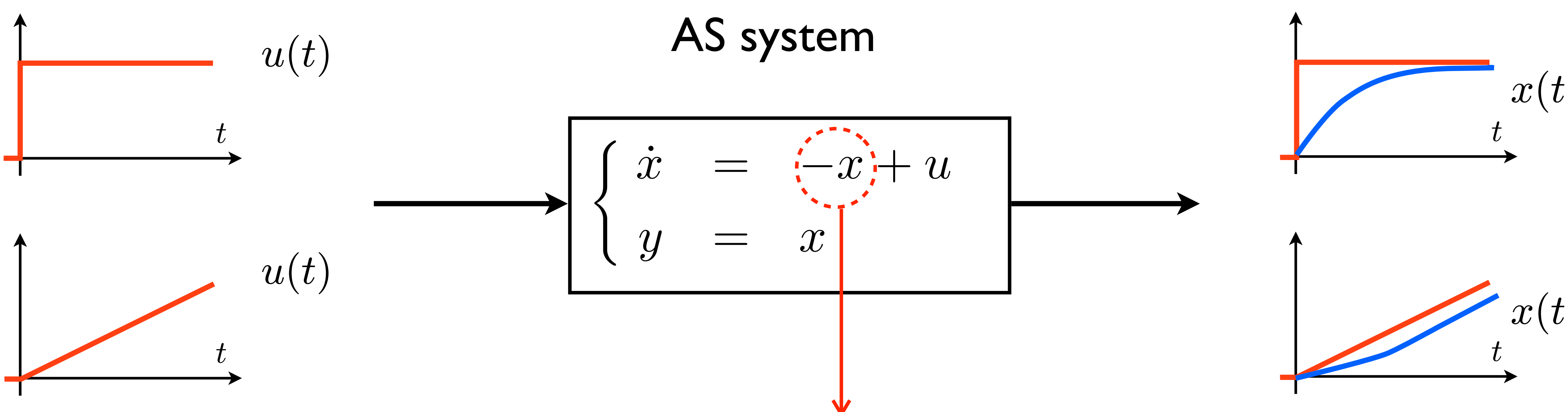


instability
at least one eigenvalue with positive real part
(the case $\mathrm{Re}(\lambda_i) = 0$ and Jordan block $\dim > 1$ is not shown)

remarks

- stability is an intrinsic characteristic of the system, depends only on A
- stability does not depend upon the applied input nor from B, C or D

example



but system is
asymptotically stable

"diverging"
response due
to the input
not to the system

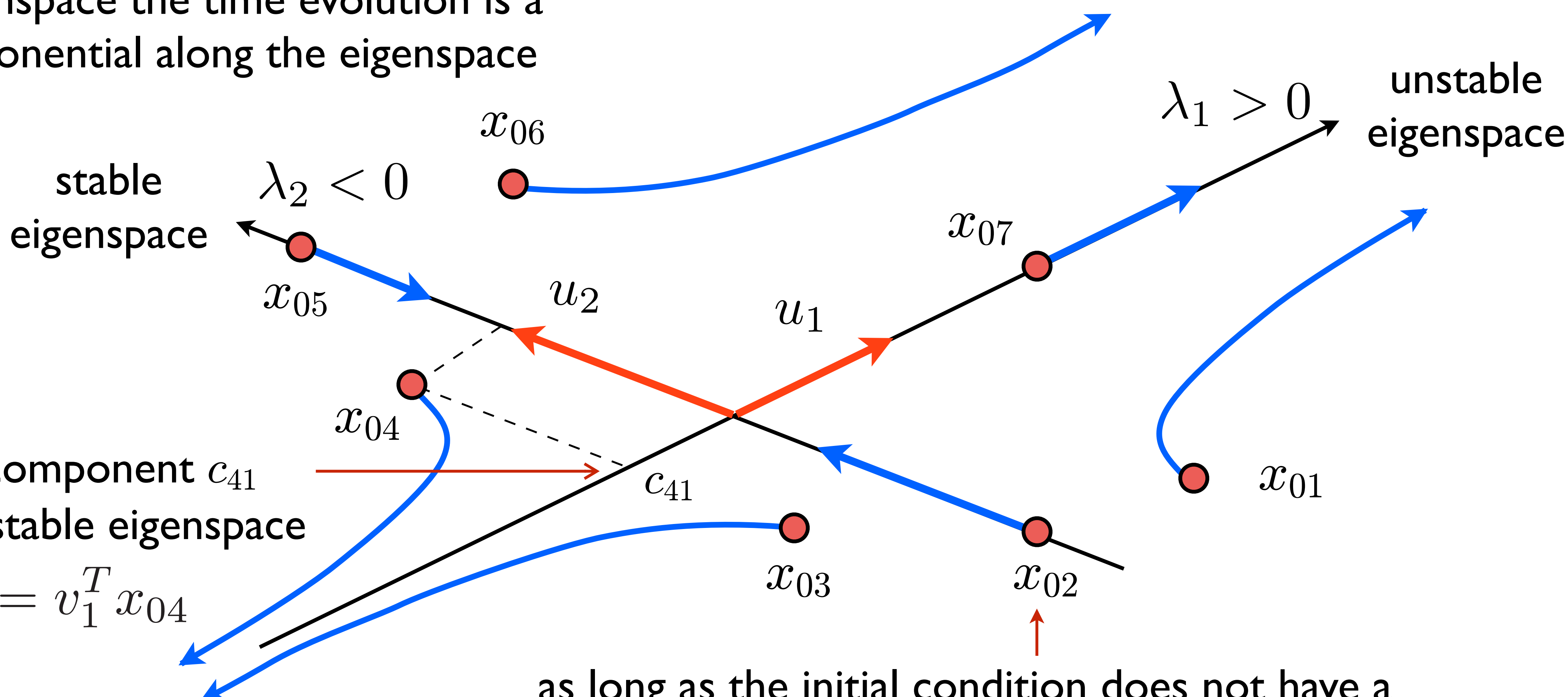
remarks

unstable systems can have bounded or converging solutions for some specific initial conditions

$$x(t) = e^{\lambda_1 t} u_1 v_1^T x_0 + e^{\lambda_2 t} u_2 v_2^T x_0$$

aperiodic modes $\lambda_1 > 0$, $\lambda_2 < 0$ → unstable system

x_{05} has no component along the unstable eigenspace the time evolution is a decaying exponential along the eigenspace



x_{04} has a component c_{41} along the unstable eigenspace

$$c_{41} u_1 = v_1^T x_{04}$$

as long as the initial condition does not have a component along the unstable eigenspace the zero input response does not diverge

remarks

- if the system is asymptotically stable then the output ZIR also converges to 0 (the converse is not true)
- if the system is (marginally) stable then the output ZIR is bounded (the converse is not true)
- if the system is unstable it does not necessarily imply that the output will diverge for some initial condition (it may never diverge)

$$y = Ce^{At}x_0 = \sum_{i=1}^{n} e^{\lambda_i t} \boxed{C\, u_i}\, v_i^T x_0$$

this term may be zero for some u_i

example

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

(compute $y_{ZIR}(t)$)

examples

- MSD with no friction and no spring $\quad m\ddot{s} = f \qquad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \lambda_1 = 0$

$$ma(\lambda_1) = 2$$

eigenspace V_1 is generated by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and therefore $mg(\lambda_1) = 1 < ma(\lambda_1)$

system in unstable (with a non-zero initial velocity, the mass will move with constant velocity and the position will grow linearly with time)

- MSD with no spring $\quad m\ddot{s} + \mu\dot{s} = f \quad A = \begin{pmatrix} 0 & 1 \\ 0 & -\mu/m \end{pmatrix} \quad \lambda_1 = 0$

$$\lambda_2 = \text{-}\mu/m < 0$$

since $ma(\lambda_1) = 1 = mg(\lambda_1)$ for the zero eigenvalue $\lambda_1 = 0$, the system is marginally stable (from a generic initial condition, the ZIR velocity will go to zero while the ZIR position will asymptotically stop at a constant value which depends upon the initial conditions)

LTI stability criterion: Routh criterion

In order to establish if a LTI system is asymptotically stable we do not need to compute the eigenvalues but just the **sign of their real parts**

generic polynomial of order n

$$p(\lambda) = a_n \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1 \lambda + a_0$$

A **necessary** condition in order for the roots of $p(\lambda) = 0$ to have all negative real part is that the coefficients need to have all the same sign

- if all the roots of $p(\lambda) = 0$ have negative real part then the coefficients have the same sign
- if a coefficient a_i is null then the coefficients do not have the same sign and therefore the necessary condition is not satisfied

Routh-Hurwitz stability criterion

In order to state a necessary and sufficient condition we need to build a table

$$p(\lambda) = a_n \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1 \lambda + a_0$$

Routh table

directly from $p(\lambda)$

row n	a_n	a_{n-2}	a_{n-4}	$\cdots$
row n-1	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
row n-2	b_1	b_2	$\cdots$	
	c_1	c_2	$\cdots$	
	d_1	$\cdots$		
row 1	$\vdots$			
row 0				

computed as

$$b_1 = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}$$

$$b_2 = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}$$

$$c_1 = -\frac{1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix}$$

$$c_2 = -\frac{1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix}$$

"missing" terms can be set to 0

- the Routh table has a finite number of elements and has $n+1$ rows
- an entire row can be multiplied by a positive number without altering the result

Routh-Hurwitz stability criterion

If the Routh table can be completed then we have the following N&S condition

All the roots of $p(\lambda) = 0$ have negative real part **iff** there are no sign changes in the first column of the Routh table

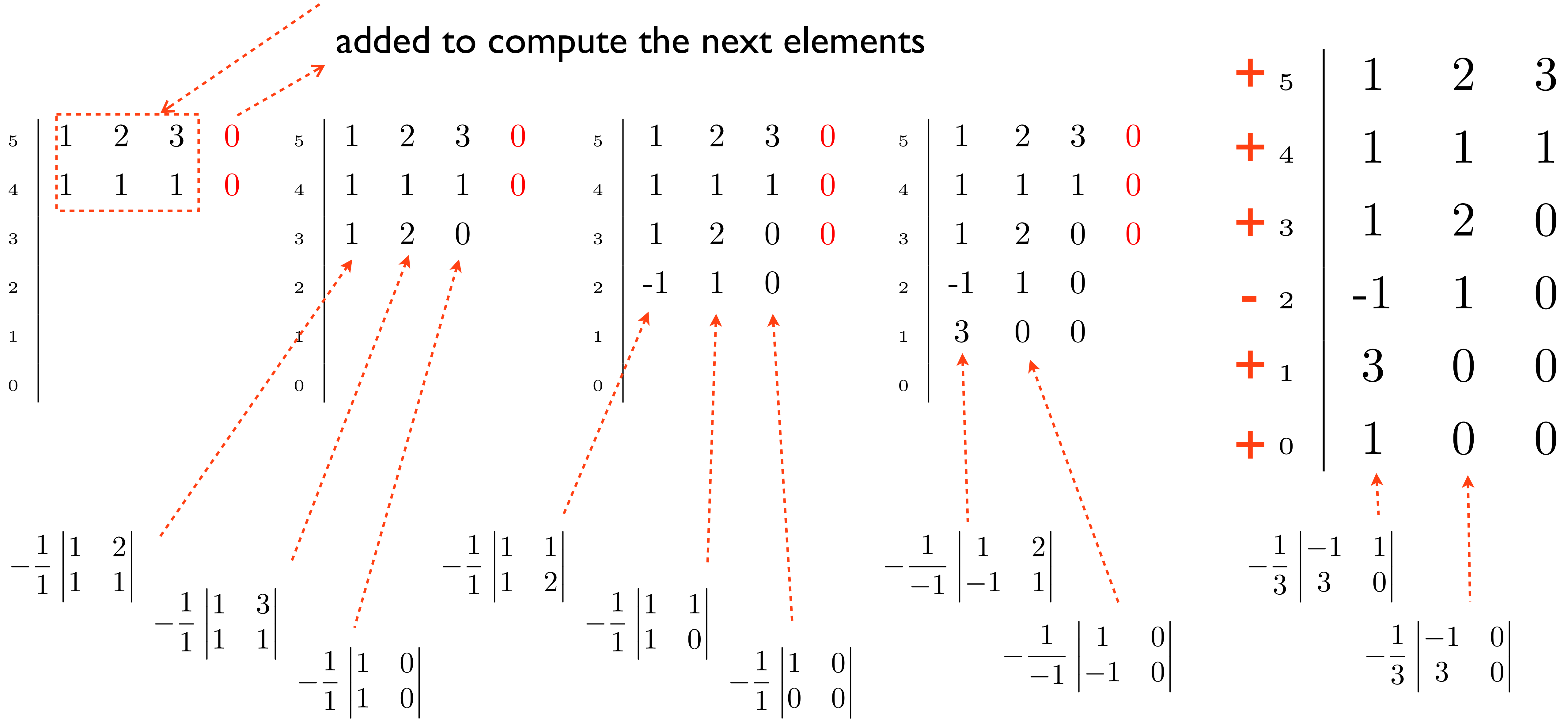
applied to the characteristic polynomial we have the following stability criterion

A LTI system is asymptotically stable **iff** the Routh table built from the characteristic polynomial has **no sign changes** in the first column

- if the table cannot be completed (due to some 0 in the first column) then **not** all the roots have negative part
- the number of sign changes in the first column of the Routh table is equal to the number of roots with positive real part

Routh table example

$$p(\lambda) = \lambda^5 + \lambda^4 + 2\lambda^3 + \lambda^2 + 3\lambda + 1$$



the table has been completed, 2 sign changes in the first column (from row 3 to row 2 and from row 2 to row 1) so 2 roots with positive real part

Routh table example

second order polynomial

$$p(\lambda) = a\lambda^2 + b\lambda + c$$

Routh table

$$\begin{array}{|cc} a & c \\ b & 0 \\ c & \end{array}$$

- for a second order polynomial, the necessary condition is also sufficient (for the 2 roots to have negative real part)
- if c has different sign than a and b, then 1 root has positive real part
- if b has different sign than a and c, then both roots have positive real part

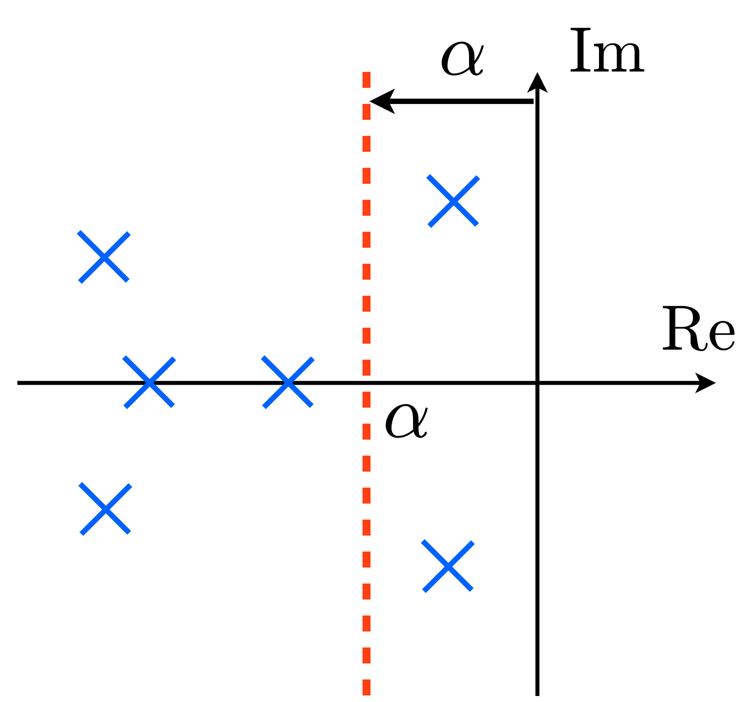
Routh table example

we want to use the Routh criterion in order to state N&S condition for the roots of a polynomial to have real part less than a given α

since $\mathrm{Re}[\lambda] < \alpha$ $\longleftrightarrow$ $\mathrm{Re}[\lambda - \alpha] < 0$ setting $\lambda - \alpha = \eta$

$\mathrm{Re}[\lambda] < \alpha$ for $p(\lambda)$	$\longleftrightarrow$	$\mathrm{Re}[\eta] < 0$ for $p(\eta) = p(\lambda)\vert_{\lambda = \eta + \alpha}$

in order to check if the roots of $p(\lambda) = 0$ all have real part smaller than α, we can apply the Routh criterion to the polynomial $p(\eta)$



this corresponds to a translation of the Im axis

from linear to nonlinear

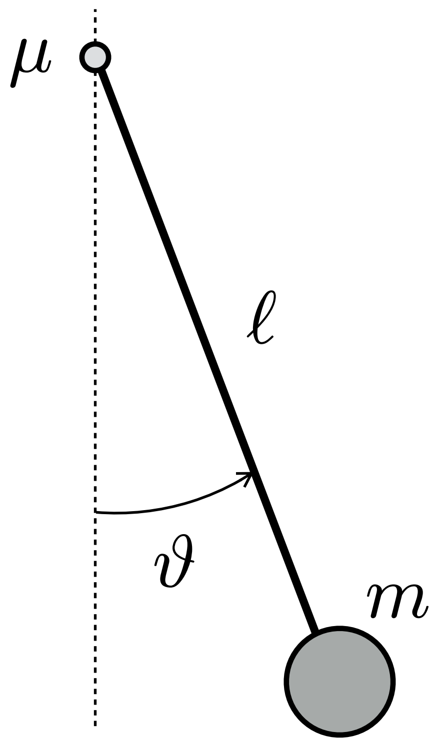
Nonlinear systems (see slides StabilityTheory by Prof. G. Oriolo):

- equilibrium points
- examples
- stable equilibrium state
- indirect method of Lyapunov

the remaining slides of Prof. Oriolo are supplementary

nonlinear systems - equilibrium states

pendulum example



$$m\,\ell^2\,\ddot{\vartheta} + m\,g\,\ell\sin\vartheta + \mu\,\dot{\vartheta} = 0$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \vartheta \\ \dot{\vartheta} \end{pmatrix} \longrightarrow \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{\ell}\sin\, x_1 - \frac{\mu}{m\,\ell^2} x_2 \end{aligned}$$

in the general form $\dot{x} = f(x)$

we are going to look for those states x_e (**equilibrium states**) for which $\dot{x} = 0$

that is for which $f(x_e) = 0$

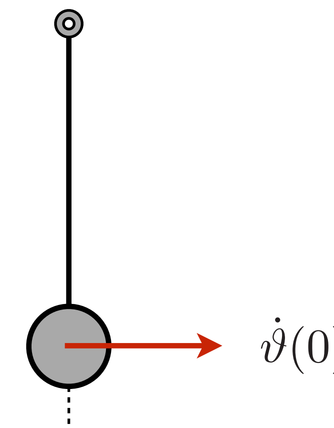
2 equilibrium states

$$x_{e1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad x_{e2} = \begin{pmatrix} \pi \\ 0 \end{pmatrix}$$

down rest position — upright rest position

solution on phase plane: pendulum

no damping case ($\mu = 0$) $$m\,\ell^2\,\ddot{\vartheta} + m\,g\,\ell\sin\vartheta = 0$$

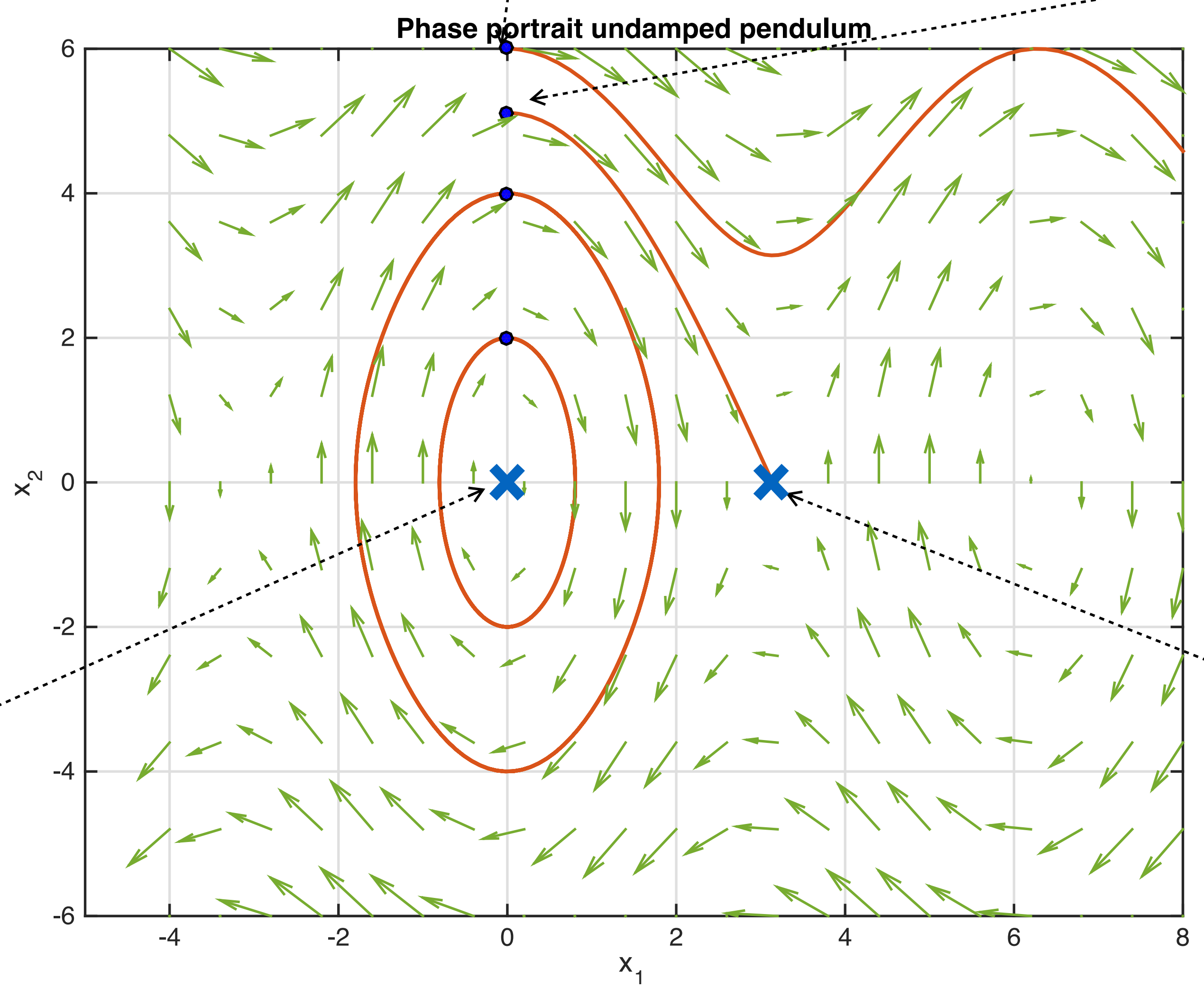


from this initial state the pendulum will never stop rotating

particular initial condition from which the pendulum reaches autonomously (asymptotically) the upright rest position

phase portrait

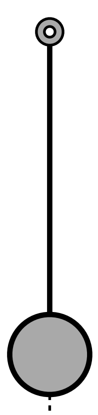
$\dot{\vartheta}(t)$

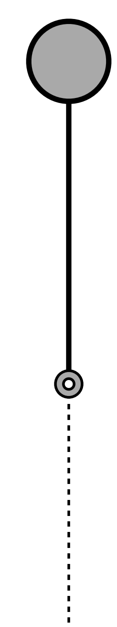


$\vartheta(t)$

x_{e1}

(stable) equilibrium state





x_{e2}

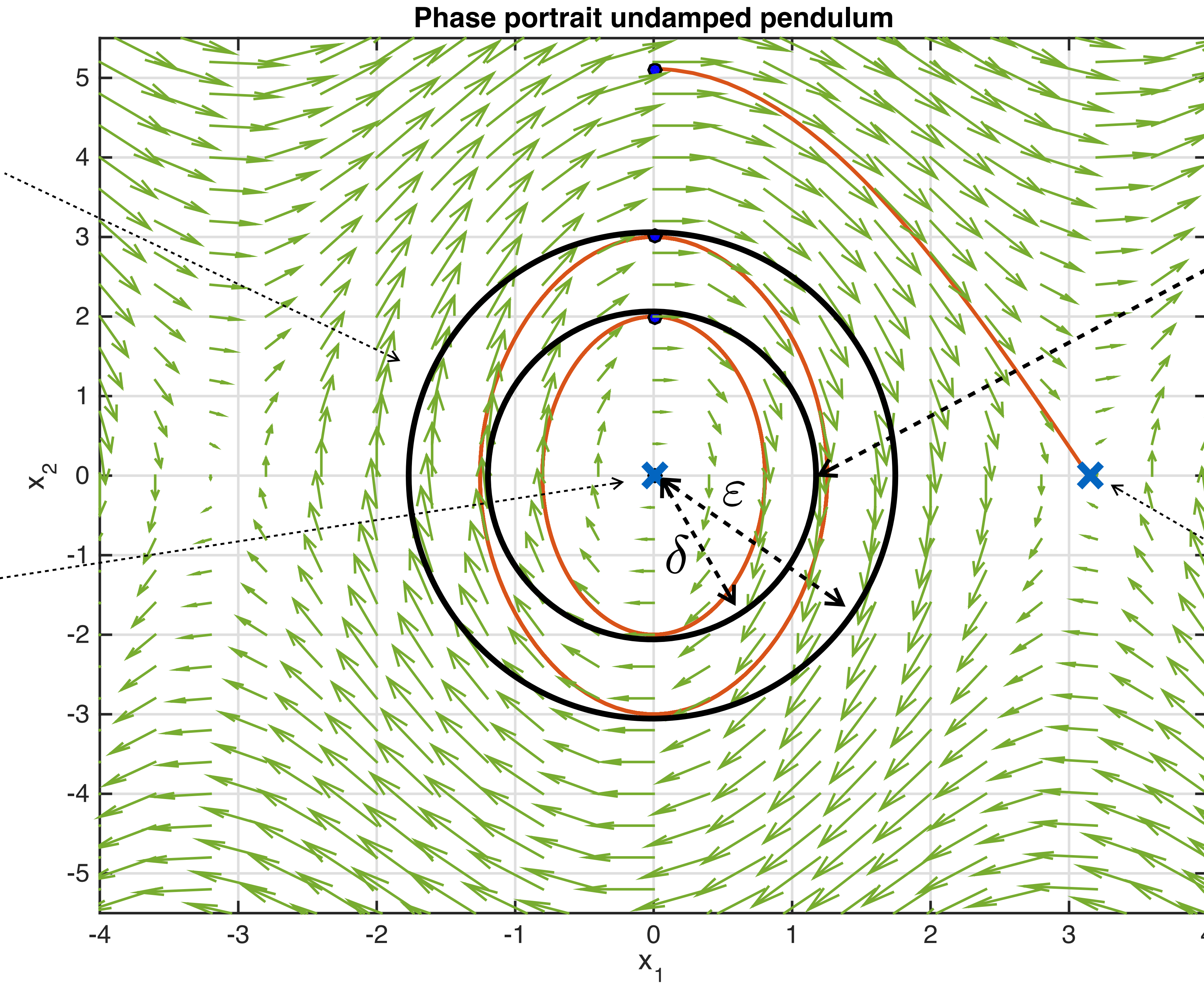
(unstable) equilibrium state

solution on phase plane: pendulum

no damping case ($\mu = 0$) $\qquad m\,\ell^2\,\ddot{\vartheta} + m\,g\,\ell\sin\vartheta = 0$

x_{e1} **stable** equilibrium state

for a given neighbourhood of radius ε of x_{e1} we can find a neighbourhood of radius δ such that the stability condition is verified



even starting on the border we remain inside ε

x_{e1} (stable) equilibrium state

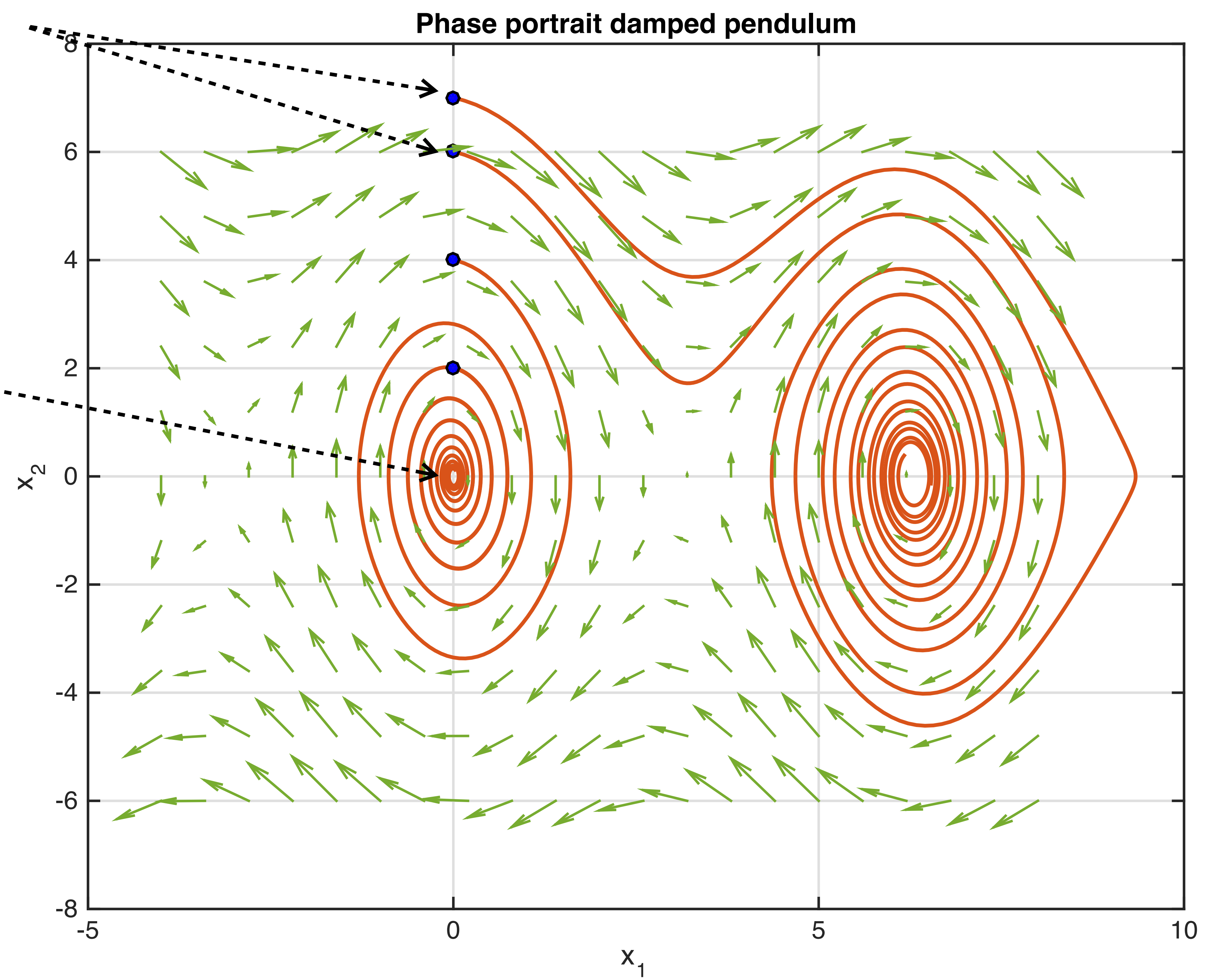
x_{e2} (unstable) equilibrium state

solution on phase plane: pendulum

solutions with non-zero damping

from these initial states the pendulum will go over the upright position to finally asymptotically stop in the down rest position

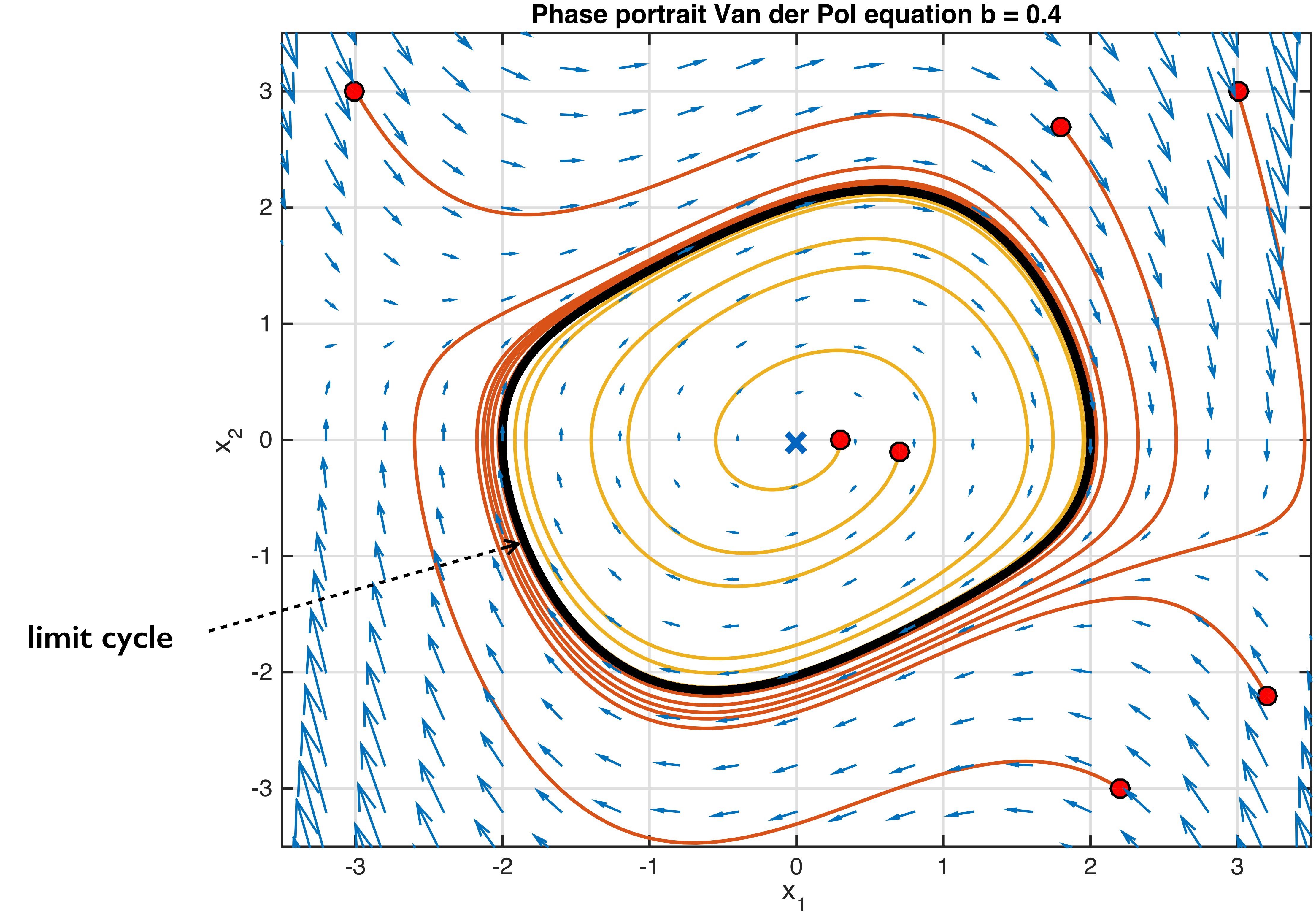
this equilibrium state x_{e1} is now asymptotically stable



solution on phase plane: Van der Pol oscillator

$$\ddot{x} - b(1 - x^2)\dot{x} + x = 0$$

is the origin stable?



solution on phase plane: Van der Pol oscillator

$$\ddot{x} - b(1 - x^2)\dot{x} + x = 0$$

for a given neighbourhood of radius ε of x_{e1} there is **no** neighbourhood of radius δ such that the stability condition is verified