Control Systems

Laplace domain analysis

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outline

- introduce the Laplace unilateral transform
- define its properties
- show its advantages in turning ODEs into algebraic equations
- define an Input/Output representation of the system through the transfer function
- explore the structure of the transfer function
- solve a realization problem

Laplace transform

real valued function in the real variable t $\quad f(t) \xrightarrow{\mathcal{L}} F(s) \quad$ complex valued function in the complex variable s

$t \in \mathbb{R}$ $\qquad$ $s \in \mathbb{C}$

$$F(s) = \int_0^\infty f(t) e^{-st} dt$$

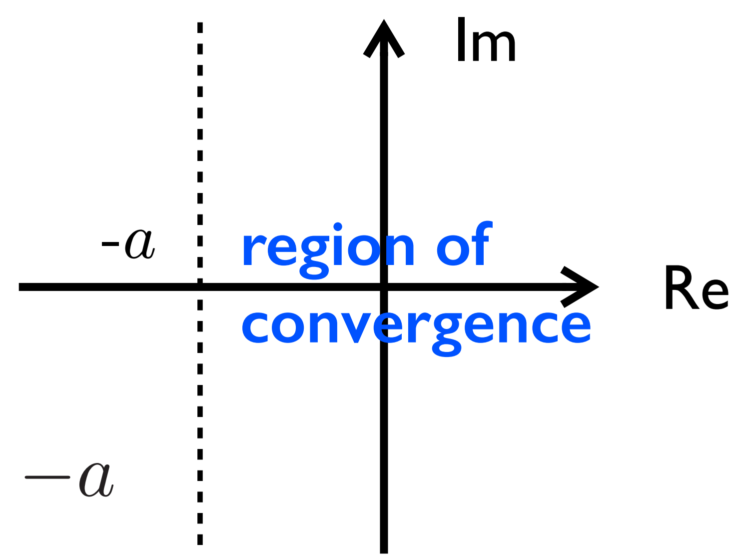
Laplace (unilateral) transform

for $f(t)$ with no impulse and no discontinuities at $t = 0$

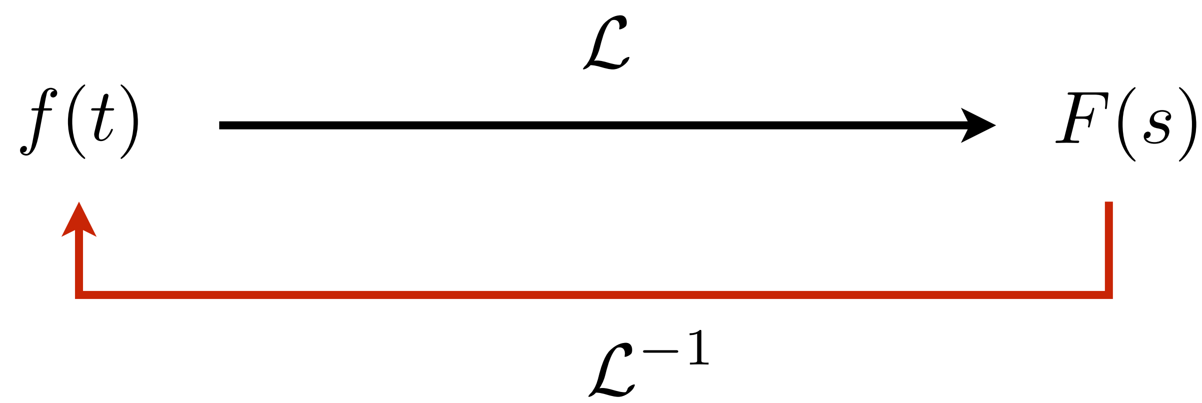
$$F(s) = \mathcal{L}[f(t)]$$

region of existence:
for all s with real part greater equal to an **abscissa of convergence** σ_0

example $\qquad f(t) = e^{-at}, \quad a > 0, \quad \sigma_0 = -a$



inverse Laplace transform

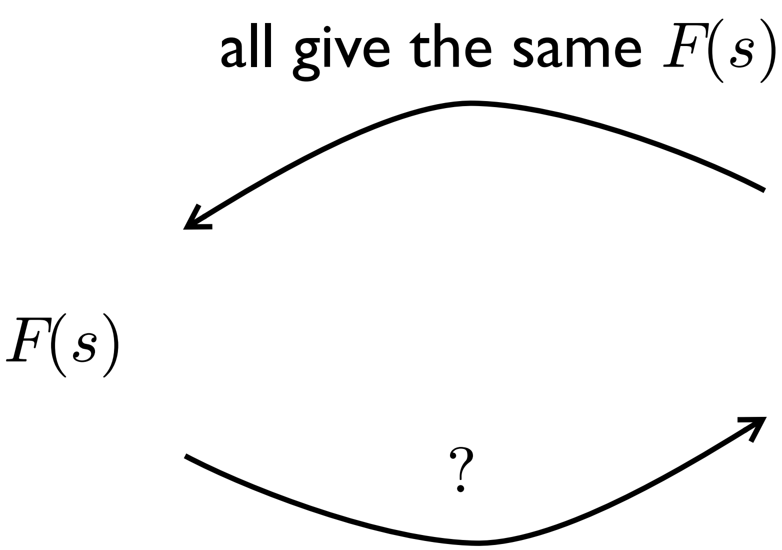


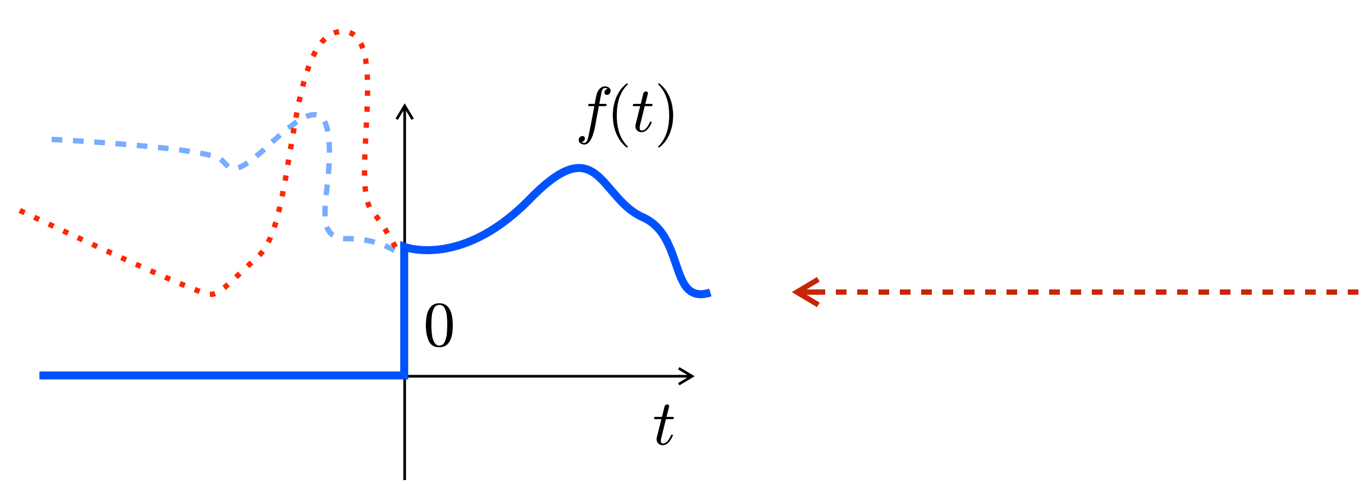
$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\alpha - j\infty}^{\alpha - j\infty} F(s)e^{st}ds$$

unique for one-sided functions $f(t)$ defined for $t \geq 0$ or $f(t) = 0 \quad \text{for} \quad t < 0$

(one-to-one)

with this assumption the inverse Laplace transform is unique





Laplace transform

Laplace transform of the **impulse**

$$F(s) = \int_{0^-}^{\infty} f(t) e^{-st} dt$$

correct definition of the Laplace transform

$$\mathcal{L}[\delta(t)] = \int_{0^-}^{\infty} \delta(t) e^{-st} dt = \int_{-\infty}^{\infty} \delta(t) e^{-st} dt = e^{-s0} = 1$$

Linearity

$$\mathcal{L}[\alpha f(t) + \beta g(t)] = \alpha \mathcal{L}[f(t)] + \beta \mathcal{L}[g(t)]$$

Derivative property

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = s\mathcal{L}[f(t)] - f(0)$$

can be applied iteratively

$$\mathcal{L}\left[\ddot{f}(t)\right] = s^2\mathcal{L}[f(t)] - sf(0) - \dot{f}(0)$$

very useful in model derivation

LTI systems

also useful here $\dot{x}(t) = Ax(t) + Bu(t) \qquad x(0) = x_0$

(proof): linearity + derivative property

algebraic solution

$$X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}B\ U(s)$$

and therefore

$$Y(s) = C(sI - A)^{-1}x_0 + \left[C(sI - A)^{-1}B + D\right]\ U(s)$$

LTI systems

state ZIR transform — state ZSR transform

$$X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}B\,U(s)$$

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

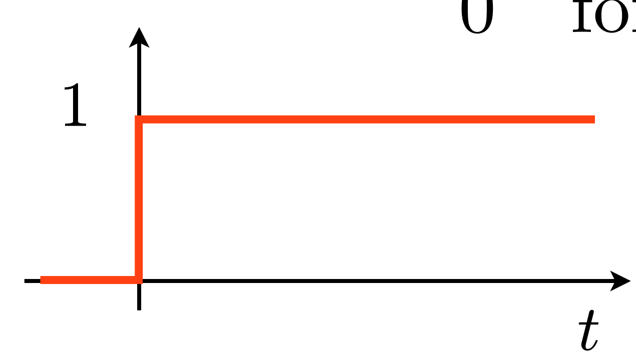
and therefore, comparing

$$\mathcal{L}\left[e^{At}\right] = (sI - A)^{-1}$$

$$\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a} \qquad \mathcal{L}\left[\delta_{-1}(t)\right] = \frac{1}{s}$$

Heaviside step function

$$\delta_{-1}(t) \quad \begin{matrix} 1 & \text{for} & t \geq 0 \\ 0 & \text{for} & t < 0 \end{matrix}$$



LTI systems

$$y(t) = Ce^{At}x_0 + \int_0^t w(t-\tau)u(\tau)d\tau$$

with $\quad w(t) = Ce^{At}B + D\,\delta(t)$

transform $\quad W(s) = C(sI - A)^{-1}B + D$

$$\begin{aligned} Y(s) &= C(sI-A)^{-1}x_0 + \left[C(sI-A)^{-1}B + D\right]\; U(s) \\ &= C(sI-A)^{-1}x_0 + W(s)\; U(s) \end{aligned}$$

general result

Convolution theorem

$$\mathcal{L}\left[\int_0^t w(t-\tau)u(\tau)d\tau\right] = W(s)\; U(s)$$

being $\quad \mathcal{L}[\delta(t)] = 1 \quad$ the output response transform corresponding to $u(t) = \delta(t)$ is indeed $W(s)$

same for $H(s)$

transfer function

Input/Output behavior

$x_0 = 0$ (ZSR)

$$y_{\mathrm{ZS}}(t) = \int_0^t w(t-\tau)u(\tau)d\tau$$

$$Y_{\mathrm{ZS}}(s) = W(s)\ U(s)$$

Transfer function

$$\begin{aligned} W(s) &= C(sI-A)^{-1}B + D \\ &= \frac{Y_{\mathrm{ZS}}(s)}{U(s)} \\ &= \mathcal{L}\left[w(t)\right] \end{aligned}$$

Input/Output behavior

independent from state choice?

independent from state space representation?

transfer function

$$z = Tx \qquad \det(T) \neq 0$$

$$(A, B, C, D) \longrightarrow (\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D})$$

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \qquad \begin{aligned} \dot{z} &= \widetilde{A}z + \widetilde{B}u \\ y &= \widetilde{C}z + \widetilde{D}u \end{aligned}$$

$$\widetilde{A} = T\,A\,T^{-1} \qquad \widetilde{B} = T\,B \qquad \widetilde{C} = C\,T^{-1} \qquad \widetilde{D} = D$$

$$W(s) = C(sI - A)^{-1}B + D = \widetilde{C}(sI - \widetilde{A})^{-1}\widetilde{B} + \widetilde{D}$$

for a given system, the transfer function is **unique**

(obvious since it's the Laplace transform of the impulsive response which is independent from the system representation)

shape of the transfer function

$$W(s) = C(sI - A)^{-1}B + D$$

inverse through the adjoint (transpose of the cofactor matrix)

$$\frac{1}{\det(sI - A)} C\,(\text{adjoint of}\,(sI - A))\,B + D$$

$$\text{cofactor}(i,j) = (-1)^{i+j}\ \text{minor}(i,j)$$ polynomial of order n - 1

polynomial of order n - 1

$$\frac{1}{\det(sI - A)} C\,(\text{cofactor of}\,(sI - A))^T\,B + D$$

polynomial of order n

→ rational function (strictly proper)

strictly proper rational function: degree of numerator < degree of denominator
proper rational function: degree of numerator = degree of denominator

we may have **cancellations of common factors** between the numerator and the denominator (final denominator degree may be less than n)

shape of the transfer function

$$W(s) = \underbrace{\text{strictly proper rational function} \ + \ D}_{\text{proper rational function}}$$

after cancellations of common terms

$$W(s) = \frac{N(s)}{D(s)}$$

$D = 0$ **strictly proper** rational function

$D \neq 0$ **proper** rational function

$$W(s) = \frac{N(s)}{D(s)} \xrightarrow{\text{roots}} \begin{array}{l}\textbf{zeros}\\ \textbf{poles}\end{array}$$

(for coprime $N(s)$ & $D(s)$)

i.e. no common roots

from previous analysis the poles are a subset of the eigenvalues of A

$$\{\text{poles}\} \subseteq \{\text{eigenvalues}\}$$

more on this later

partial fraction expansion (distinct roots case)

Let $F(s) = \dfrac{N(s)}{D(s)}$ be a **strictly proper rational** function with **coprime** $N(s)$ and $D(s)$ and **distinct roots** of $D(s)$

i.e. $$D(s) = a_n \prod_{i=1}^{n} (s - p_i) \quad \Rightarrow \quad F(s) = \frac{N(s)}{a_n \prod_{i=1}^{n} (s - p_i)}$$

then $F(s)$ can be expanded as

$$F(s) = \sum_{i=1}^{n} \frac{R_i}{s - p_i}$$

with the **residues** R_i computed as

$$R_i = \left[(s - p_i) F(s) \right] \Big|_{s=p_i}$$

this result will be used for the

- transfer function $W(s)$
- output zero-state response $Y(s)$

poles & eigenvalues

$$W(s) = \frac{N(s)}{D(s)} \quad \xleftarrow{\ ?\ } \quad \text{from def} \quad W(s) = \frac{1}{\det(sI - A)} N'(s)$$

$\det(sI - A)$ ← characteristic polynomial

if $\det(sI - A)$ and $N'(s)$ coprime {poles} = {eigenvalues}

if $\det(sI - A)$ and $N'(s)$ not coprime {poles} subset of {eigenvalues}

here visible → Input/Output representation (poles)

here visible → Input/State/Output representation (eigenvalues)

Input/Output representation vs Input/State/Output representation

Transfer function | **State space**

we need to understand when & why this happens
(so to understand when we can consider the transfer function equivalent to a state space representation)

poles & eigenvalues

(distinct eigenvalues of A case)

n = state space dimension = dimension of A = number of eigenvalues

n_p = number of poles in $W(s)$

$D = 0$ case

partial fraction expansion

$$W(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{\prod_{i=1}^{n_p}(s-p_i)} = \sum_{i=1}^{n_p} \frac{R_i}{s-p_i}$$

done the same analysis in t for $n = 2$

$$W(s) = \mathcal{L}[w(t)] = \mathcal{L}[Ce^{At}B] = \mathcal{L}[C\left(\sum_{j=1}^{n} e^{\lambda_j t} u_j v_j^T\right)B] = \sum_{j=1}^{n} \frac{Cu_j \; v_j^T B}{s-\lambda_j}$$

spectral form

if
$v_j^T B = 0$
and/or
$Cu_j = 0$
the **eigenvalue** λ_j does **not** appear as a **pole**

we have a "**hidden mode**" associated to the eigenvalue λ_j
(see structural properties)

NB distinct eigenvalues is different from diagonalizable

poles & eigenvalues (distinct eigenvalues of A case)

fact: all the natural modes appear in the state ZIR for some generic initial condition → e^{At}

If for an eigenvalue λ_j we have that

$v_j^T B = 0$ implies the corresponding mode will not appear in the state impulsive response → $e^{At}B = H(t)$

the corresponding mode (or eigenvalue) is said to be **uncontrollable**

$Cu_j = 0$ implies the corresponding mode will not appear in the output transition matrix → $Ce^{At} = \Psi(t)$

the corresponding mode (or eigenvalue) is said to be **unobservable**

poles & eigenvalues

(distinct eigenvalues of A case)

Theorem

Every pole is an eigenvalue.

An eigenvalue λ_i becomes a pole if and only if it is both controllable and observable

$$v_i^T B \neq 0 \qquad \text{and} \qquad C u_i \neq 0$$

or equivalently the following two PBH rank tests are both verified

$$\text{rank}\left(A - \lambda_i I \;\middle|\; B \right) = n \qquad \text{and} \qquad \text{rank}\begin{pmatrix} A - \lambda_i I \\ \hline C \end{pmatrix} = n$$

Popov-Belevitch-Hautus controllability test

Popov-Belevitch-Hautus observability test

(the PBH test could be tested for a generic λ but matrix A - λI loses rank only for $\lambda = \lambda_i$)

poles & eigenvalues

(distinct eigenvalues of A case)

Where does the PBH test comes from?

Observability (sketch):

$$\text{rank}\begin{pmatrix} A-\lambda_i I \\ \hline C \end{pmatrix} < n$$

means that the rectangular $(n+1)$ x n matrix has not full column rank and therefore it has a non-zero nullspace, that is there exists a n vector u_i such that

$$\begin{pmatrix} A-\lambda_i I \\ \hline C \end{pmatrix} u_i = 0 \quad \longleftrightarrow \quad \begin{cases} (A-\lambda_i I)\, u_i = 0 \\ C u_i = 0 \end{cases} \quad \longleftrightarrow \quad \begin{cases} A u_i = \lambda_i u_i \\ C u_i = 0 \end{cases}$$

that is there exists an eigenvector which belongs to the nullspace of C (or the corresponding mode is unobservable)

example

$$A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$(sI - A)^{-1} = \begin{pmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s-1)} \\ 0 & \frac{1}{s-1} \end{pmatrix} = M_1 \frac{1}{s+1} + M_2 \frac{1}{s-1}$$

fractional decomposition works also for rational matrices

with $$M_1 = \left[(s+1)(sI - A)^{-1}\right]_{s=-1} = \begin{pmatrix} 1 & -1/2 \\ 0 & 0 \end{pmatrix}$$

$$M_2 = \left[(s-1)(sI - A)^{-1}\right]_{s=1} = \begin{pmatrix} 0 & 1/2 \\ 0 & 1 \end{pmatrix}$$

a different way to compute the matrix exponential

$$e^{At} = M_1 e^{-t} + M_2 e^{t}$$

both natural modes appear (as it should be) in the state transition matrix

example

$$(sI - A)^{-1}B = \begin{pmatrix} \frac{1}{s+1} \\ 0 \end{pmatrix}$$ → mode corresponding to λ_2 does not appear ← $e^{At}B$

$$C(sI - A)^{-1} = \begin{pmatrix} 0 & \frac{1}{s-1} \end{pmatrix}$$ → mode corresponding to λ_1 does not appear ← Ce^{At}

$$W(s) = 0$$

no poles

$$w(t) = Ce^{At}B = 0$$

forced response will always be zero independently from the input applied (look at the 2 first order ODE)

equivalently

$u_1 \quad / \quad (A - \lambda_1 I)u_1 = 0 \qquad \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} u_1 = 0 \qquad u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad v_1^T = \begin{bmatrix} 1 & -1/2 \end{bmatrix}$

$u_2 \quad / \quad (A - \lambda_2 I)u_2 = 0 \qquad \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} u_2 = 0 \qquad u_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad v_2^T = \begin{bmatrix} 0 & 1/2 \end{bmatrix}$

$$v_1^T B \neq 0 \quad v_2^T B = 0 \qquad\qquad Cu_1 = 0 \quad Cu_2 \neq 0$$

example

equivalently with the PBH rank test

controllability test

mode corresponding to λ_1 is controllable

$$\text{rank}\left(\begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 2 & 0 \end{array}\right) = 2 = n$$

mode corresponding to λ_2 is uncontrollable

$$\text{rank}\left(\begin{array}{cc|c} -2 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right) = 1 < n$$

observability test

$$\text{rank}\left(\begin{array}{cc} 0 & 1 \\ 0 & 2 \\ \hline 0 & 1 \end{array}\right) = 1 < n$$

$$\text{rank}\left(\begin{array}{cc} -2 & 1 \\ 0 & 0 \\ \hline 0 & 1 \end{array}\right) = 2 = n$$

mode corresponding to λ_1
is unobservable

mode corresponding to λ_2
is observable

poles & eigenvalues (general case)

Theorem

Every pole is an eigenvalue.

An eigenvalue λ_i becomes a pole with the multiplicity m_a (algebraic multiplicity) if and only if both PBH rank tests are verified

$$\text{rank}\left(A - \lambda_i I \mid B \right) = n$$

controllability

$$\text{rank}\begin{pmatrix} A - \lambda_i I \\ \hline C \end{pmatrix} = n$$

observability

NB - If one of the two conditions is not verified then the eigenvalue λ_i will appear as a pole with multiplicity strictly less than the algebraic multiplicity, possibly even 0 (in this case we will have a hidden eigenvalue). In particular the eigenvalue will appear **at most** as a pole with multiplicity equal to its **index** (dimension of the largest Jordan block).

NB - If for an eigenvalue λ_i the geometric $mg(\lambda_i) > 1$ then there exists a hidden dynamics.

example

$$A_0 = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

PBH rank test verified for $B = \begin{bmatrix} \forall \\ \forall \\ \neq 0 \end{bmatrix}$ $C = \begin{bmatrix} \neq 0 & \forall & \forall \end{bmatrix}$

→ easily seen from → $A_0 - \lambda_1 I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$$(sI - A_0)^{-1} = \frac{1}{(s-\lambda_1)^3} \begin{bmatrix} (s-\lambda_1)^2 & (s-\lambda_1) & 1 \\ 0 & (s-\lambda_1)^2 & (s-\lambda_1) \\ 0 & 0 & (s-\lambda_1)^2 \end{bmatrix}$$

$B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $C_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ → $F_1(s) = \dfrac{1}{(s-\lambda_1)^3}$ ← index of λ_1

$B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $C_2 = C_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ → $F_2(s) = \dfrac{(s-\lambda_1)^2}{(s-\lambda_1)^3} = \dfrac{1}{s-\lambda_1}$

$B_3 = B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $C_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ → $F_3(s) = 0$

example

$$A_4 = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix} \quad \text{since} \quad A_4 - \lambda_1 I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

the PBH rank test will never be satisfied independently from B and C. At most the eigenvalue will appear as a pole with multiplicity = index of λ_1 = 2

$$(sI - A_4)^{-1} = \frac{1}{(s-\lambda_1)^3} \begin{bmatrix} (s-\lambda_1)^2 & (s-\lambda_1) & 0 \\ 0 & (s-\lambda_1)^2 & 0 \\ 0 & 0 & (s-\lambda_1)^2 \end{bmatrix} = \frac{1}{(s-\lambda_1)^2} \begin{bmatrix} (s-\lambda_1) & 1 & 0 \\ 0 & (s-\lambda_1) & 0 \\ 0 & 0 & (s-\lambda_1) \end{bmatrix}$$

$$B_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad C_4 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \qquad F_4(s) = \frac{1}{(s-\lambda_1)^2}$$

$$B_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad C_5 = C_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \qquad F_5(s) = \frac{s-\lambda_1}{(s-\lambda_1)^2} = \frac{1}{s-\lambda_1}$$

$$B_6 = B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad C_6 = C_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \qquad F_6(s) = 0$$

example

$$A_7 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix} \qquad (sI - A_7)^{-1} = \frac{1}{s - \lambda_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

etc ...

the PBH rank test will never be satisfied independently from B and C. At most the eigenvalue will appear as a pole with multiplicity $=$ index of $\lambda_1 = 1$

example

is the system characterized by the transfer function $\dfrac{1}{s^2}$ stable?

- when we start from the transfer function, we implicitly assume that the eigenvalues (and their algebraic multiplicity) coincide with the poles

fundamental assumption

- we have seen that an eigenvalue appears as a pole with multiplicity at most equal to its index (dimension of its larger Jordan block)
- for the pole multiplicity to be equal to both the index and the algebraic multiplicity of the eigenvalue, there must be only one Jordan block

thus the system has the eigenvalue in $\lambda = 0$ with geometric multiplicity $= 1$ and index $= 2$

system is **unstable**

can also be seen by computing the impulse response $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\right\} = t\,\delta_{-1}(t)$

similarly for $\dfrac{1}{(s^2+1)^2}$

Laplace transform table

$\delta(t)$	1
$\delta_{-1}(t)$	$\dfrac{1}{s}$
e^{at}	$\dfrac{1}{s-a}$
$\dfrac{t^k}{k!}$	$\dfrac{1}{s^{k+1}}$
$\dfrac{t^k}{k!}e^{at}$	$\dfrac{1}{(s-a)^{k+1}}$

$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2} = \dfrac{1/2j}{s-j\omega} - \dfrac{1/2j}{s+j\omega}$
$\cos \omega t$	$\dfrac{s}{s^2+\omega^2} = \dfrac{1/2}{s-j\omega} + \dfrac{1/2}{s+j\omega}$
$\sin(\omega t + \varphi)$	$\dfrac{s \sin \varphi + \omega \cos \varphi}{s^2+\omega^2}$
$e^{at} \sin \omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$
$e^{at} \cos \omega t$	$\dfrac{(s-a)}{(s-a)^2+\omega^2}$

realizations

$$(A, B, C, D) \xrightarrow{\text{ok}} W(s) = C(sI - A)^{-1}B + D$$

how ?

infinite solutions

$$W(s) = C(sI - A)^{-1}B + D \xrightarrow{\text{realization}} (A, B, C, D)$$

- state dimension ?
- how can we easily find one state space representation $(A,\ B,\ C,\ D)$?
- may be complicated for MIMO systems (here SISO)
- we see only one, obtainable directly from the coefficients of the transfer function (others are obtainable by simple similarity transformations) with state dimension $= n$ (i.e. a **minimal realization**)

realizations

given $W(s) = \dfrac{N(s)}{D(s)}$ with $N(s)$ & $D(s)$ **coprime**

- first we determine D if $W(s)$ is proper

strictly proper

a non-zero D leads to $W(s)$ proper

$$W(s) = \boxed{\frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0}} + \boxed{D}$$

- the state has dimension n and therefore the dynamic matrix is n x n
- one possible choice for A, B, C (D has already been determined) is

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \qquad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C_c = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-1} \end{bmatrix}$$

controller canonical form
(useful for eigenvalue assignment)

realizations

the matrix A_c is called a **companion matrix** and has as characteristic polynomial

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \longrightarrow p_{A_c}(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

the pair $(A_c,\ B_c)$ has, by construction, all its natural modes **controllable**

we then say that the pair $(A_c,\ B_c)$ is, by construction, **controllable**

recall that the poles of the transfer function are also eigenvalues and therefore the matrix A_c of the realization has the minimum necessary number of eigenvalues

realizations (examples)

$$P(s) = \frac{1}{3s^2+2s+6} = \frac{\frac{1}{3}}{s^2+\frac{2}{3}s+2}$$

$$\longrightarrow \quad A_c = \begin{bmatrix} 0 & 1 \\ -2 & -2/3 \end{bmatrix} \qquad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C_c = \begin{bmatrix} 1/3 & 0 \end{bmatrix} \qquad D_c = 0$$

$$P(s) = \frac{s^2+3}{2s^2+6s+2} = \frac{s^2+3}{2(s^2+3s+1)} = \frac{1}{2}\frac{s^2+3}{s^2+3s+1} = \frac{1}{2}\left(1 + \frac{-3s}{s^2+3s+1}\right) = \frac{-3/2s}{s^2+3s+1} + \frac{1}{2}$$

$$\longrightarrow \quad A_c = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} \qquad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C_c = \begin{bmatrix} 0 & -3/2 \end{bmatrix} \qquad D_c = \frac{1}{2}$$

$$P(s) = \frac{2s^4+13s^3+6s^2+6s+7}{s^4+5s^3+3s^2+2s+1} = \frac{3s^3+2s+5}{s^4+5s^3+3s^2+2s+1} + 2$$

$$\longrightarrow \quad A_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2 & -3 & -5 \end{bmatrix} \qquad B_c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad C_c = \begin{bmatrix} 5 & 2 & 0 & 3 \end{bmatrix} \qquad D_c = 2$$