Control Systems

System response

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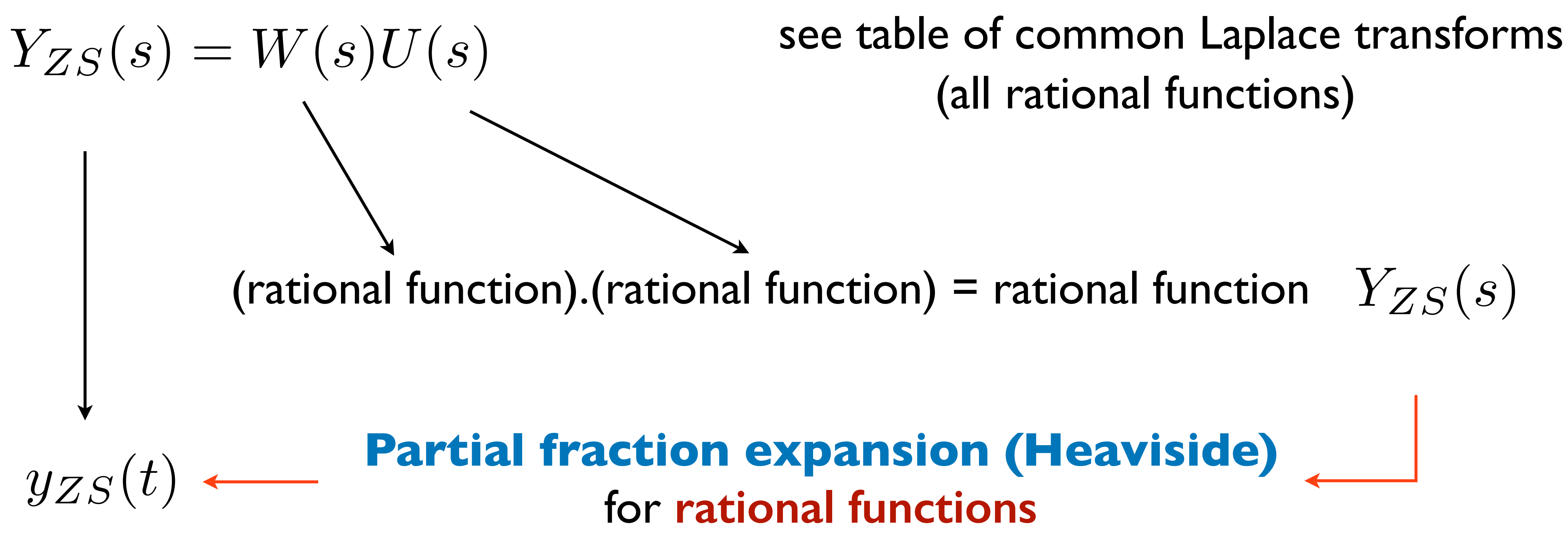
outline

- how to compute in the s-domain the forced response (zero-state response) using the transfer function
- how to inverse transform the resulting responses so to obtain the time response
- partial fraction expansion (general case)
- integral theorem for the Laplace transform
- time shifting theorem for the Laplace transform
- transfer function of a delay
- computing the forced response to a shifted input

Zero State Response (forced response)



$Y_{ZS}(s) = W(s)U(s)$

see table of common Laplace transforms
(all rational functions)

(rational function).(rational function) = rational function $Y_{ZS}(s)$

$y_{ZS}(t)$

Partial fraction expansion (Heaviside)
for **rational functions**

Basic idea: write the function, we want to find the inverse Laplace transform of, as a linear combination of "easy to transform" terms (e.g., one present in the table of transforms) and then use the linearity property of the inverse transformation

For rational functions: represent a complicated fraction as the sum of simpler fractions for which we know the inverse Laplace transform (partial fraction expansion or decomposition)

Laplace transform table

$\delta(t)$	1
$\delta_{-1}(t)$	$\dfrac{1}{s}$
e^{at}	$\dfrac{1}{s-a}$
$\dfrac{t^k}{k!}$	$\dfrac{1}{s^{k+1}}$
$\dfrac{t^k}{k!}e^{at}$	$\dfrac{1}{(s-a)^{k+1}}$

$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2} = \dfrac{1/2j}{s-j\omega} - \dfrac{1/2j}{s+j\omega}$
$\cos \omega t$	$\dfrac{s}{s^2+\omega^2} = \dfrac{1/2}{s-j\omega} + \dfrac{1/2}{s+j\omega}$
$\sin(\omega t + \varphi)$	$\dfrac{s \sin\varphi + \omega\cos\varphi}{s^2+\omega^2}$
$e^{at}\sin \omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$
$e^{at}\cos \omega t$	$\dfrac{(s-a)}{(s-a)^2+\omega^2}$

partial fraction expansion: distinct roots case

Let $F(s) = \dfrac{N(s)}{D(s)}$ be a **strictly proper** rational function with **coprime** $N(s)$ and $D(s)$ and with **distinct roots** of $D(s)$

i.e., we write $D(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_s + a_o$

as $D(s) = a_n \prod_{i=1}^{n} (s - p_i) \quad \Rightarrow \quad F(s) = \dfrac{N(s)}{a_n \prod_{i=1}^{n} (s - p_i)}$

then $F(s)$ can be expanded as

$$\boxed{F(s) = \sum_{i=1}^{n} \frac{R_i}{s - p_i}}$$

partial fraction expansion

where the **residues** R_i are computed as

$$\boxed{R_i = [(s - p_i) F(s)] \Big|_{s = p_i}}$$

- if $F(s)$ is **proper**, rewrite it as the sum of a strictly proper rational function and a constant and then expand in partial fractions only the strictly proper part

partial fraction expansion

exercises:

- given the transfer function $W(s)$ of a system S, find its impulse response $w(t)$

$$W(s) = \frac{s+2}{s(s+1)(s+10)}$$

(sol: just expand and inverse transform)

- given the dynamic matrix A, find the matrix exponential using the partial fraction expansion

$$A = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} \longrightarrow \quad e^{A\,t} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}$$

being $(sI - A)^{-1} = \begin{pmatrix} s+1 & -1 \\ 0 & s-2 \end{pmatrix}^{-1} = \dfrac{1}{(s+1)(s-2)} \begin{pmatrix} s-2 & 1 \\ 0 & s+1 \end{pmatrix}$

we expand $(sI - A)^{-1} = \Phi(s) = \dfrac{1}{s+1} R_1 + \dfrac{1}{s-2} R_2$

with $R_1 = ((s+1)\Phi(s))\Big|_{s=-1} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \qquad R_2 = ((s-2)\Phi(s))\Big|_{s=2} = \begin{pmatrix} 0 & 1/3 \\ 0 & 1 \end{pmatrix}$

and inverse transform

partial fraction expansion - general case

Let $F(s) = \dfrac{N(s)}{D(s)}$ be a **strictly proper** rational function with **coprime** $N(s)$ and $D(s)$.

Let $D(s)$ have m roots each with multiplicity n_i that is $\displaystyle\sum_{i=1}^{m} n_i = n$

then $F(s)$ can be expanded as

$$\boxed{F(s) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{R_{ij}}{(s-p_i)^j}}$$

with **residues** R_{ij} computed as

$$\boxed{R_{ij} = \left[\frac{1}{(n_i - j)!} \frac{d^{n_i - j}}{ds^{n_i - j}} \left\{ (s-p_i)^{n_i} F(s) \right\} \right]_{s=p_i}}$$

Example: verify that since $\dfrac{s-1}{s^3} = \dfrac{s}{s^3} - \dfrac{1}{s^3} = \dfrac{1}{s^2} - \dfrac{1}{s^3}$ you apply well the residue formula

and obtain $R_{11} = 0$, $R_{12} = 1$ and $R_{13} =$ -1.

partial fraction expansion

example:

find the zero-state output response (or output forced response) of the system characterized by the transfer function $W(s)$ to the input $u(t) = t$ (remember the function $u(t)$ is assumed to be zero for $t < 0$)

$$W(s) = \frac{s-1}{(s+1)(s+10)}$$

$$U(s) = \mathcal{L}[u(t)] = \frac{1}{s^2} \quad \Rightarrow \quad Y(s) = W(s)U(s) = \frac{s-1}{(s+1)(s+10)}\,\frac{1}{s^2}$$

$$Y(s) = \frac{R_{11}}{s} + \frac{R_{12}}{s^2} + \frac{R_2}{s+1} + \frac{R_3}{s+10}$$

$$\begin{aligned}
R_{11} &= \left[\frac{1}{(2-1)!}\frac{d^{2-1}}{ds^{2-1}}\left\{s^2 Y(s)\right\}\right]_{s=0} = \left[\frac{d}{ds}\left\{\frac{s-1}{(s+1)(s+10)}\right\}\right]_{s=0} = \frac{21}{100} \\
R_{12} &= \left[\frac{1}{(2-2)!}\frac{d^{2-2}}{ds^{2-2}}\left\{s^2 Y(s)\right\}\right]_{s=0} = \left[\frac{s-1}{(s+1)(s+10)}\right]_{s=0} = -\frac{1}{10} \\
R_2 &= \left[(s+1)Y(s)\right]_{s=-1} = \left[\frac{s-1}{s^2(s+10)}\right]_{s=-1} = -\frac{2}{9} \\
R_3 &= \left[(s+10)Y(s)\right]_{s=-10} = \left[\frac{s-1}{s^2(s+1)}\right]_{s=-10} = \frac{11}{900}
\end{aligned}$$

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \left(R_{11} + R_{12}t + R_2 e^{-t} + R_3 e^{-10t}\right)\delta_{-1}(t)$$

partial fraction expansion

example:

system $F(s) = \dfrac{s-1}{s(s+1)(s-10)}$ input $U(s) = \dfrac{1}{s}$

$$Y(s) = \frac{s-1}{s^2(s+1)(s-10)} \longrightarrow Y(s) = \frac{R_{11}}{s} + \frac{R_{12}}{s^2} + \frac{R_2}{s+1} + \frac{R_3}{s+10}$$

↑

this behavior is not present in both
the transfer function and the input

↓

new time behavior t appears in the output!

$$\frac{1}{s} \longrightarrow \delta_{-1}(t)$$

$$\frac{1}{s+1} \longrightarrow e^{-t}\delta_{-1}(t)$$

$$\frac{1}{s-10} \longrightarrow e^{10\,t}\delta_{-1}(t)$$

these behaviors are present in either
the input or the impulse response (or both)

partial fraction expansion

special case: $F(s)$ has k poles in $s = 0$ (generic rational function)

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{s^k D'(s)}$$

← (we isolated the roots in $s = 0$)

then

$$F(s) = \frac{R_{11}}{s} + \frac{R_{12}}{s^2} + \cdots + \boxed{\frac{R_{1k}}{s^k}} + \sum \sum \frac{R_{ij}}{(s-p_i)^j}$$

with

$$R_{1k} = \left[\frac{1}{(k-k)!} \frac{d^{k-k}}{ds^{k-k}} \left\{ s^k F(s) \right\} \right] \Big|_{s=0} = \left[s^k F(s) \right] \Big|_{s=0} = \frac{N(0)}{D'(0)}$$

leading coefficient

N.B. this result will be useful for the steady-state response to an order k input $\dfrac{t^k}{k!}$

and R_{1k} will coincide with the system gain

partial fraction expansion

for complex poles

$$\begin{cases} p = \alpha + j\beta & \xrightarrow{\text{residue}} \quad R = a + jb \\ p^* = \alpha - j\beta & \xrightarrow{\qquad\quad} \quad R^* \ \text{ complex conjugate of } R \end{cases}$$

general case

$$\longrightarrow \quad \frac{R}{(s-p)^k} + \frac{R^*}{(s-p^*)^k} = \frac{As+B}{((s-\alpha)^2+\beta^2)^k}$$

partial fraction expansion

for $k = 1$ find the residue $R = a + jb$ associated to p = $\alpha + j\beta$ (and R^*)

$$\frac{R}{s-p} + \frac{R^*}{s-p^*} = \frac{R(s-p^*) + R^*(s-p)}{(s-\alpha)^2+\beta^2} = \frac{s(R+R^*) - (Rp^* + R^*p)}{(s-\alpha)^2+\beta^2}$$

$$= \frac{s(R+R^*) - \alpha(R+R^*) - j\beta(R^*-R)}{(s-\alpha)^2+\beta^2}$$

$$R + R^* = 2a, \qquad R^* - R = -2jb \qquad\qquad A = 2a \qquad B = -2(a\alpha + b\beta)$$

$$\frac{R}{s-p} + \frac{R^*}{s-p^*} = \frac{2as - 2(a\alpha + b\beta)}{(s-\alpha)^2+\beta^2} = \frac{As+B}{(s-\alpha)^2+\beta^2}$$

rewritten as

with $As + B = A(s-\alpha) + \beta(A\alpha + B)/\beta$

$$\frac{R}{s-p} + \frac{R^*}{s-p^*} = \frac{A(s-\alpha)}{(s-\alpha)^2+\beta^2} + \frac{\beta(A\alpha+B)/\beta}{(s-\alpha)^2+\beta^2}$$

so that we recognise two known transforms

in t

$$e^{\alpha t}\left[A\cos\beta t + (A\alpha + B)/\beta \sin\beta t\right]$$

partial fraction expansion

example:

$$H(s) = \frac{1}{(s^2+1)(s-2)^2} \qquad\qquad H(s) = \frac{R_{11}}{s-2} + \frac{R_{12}}{(s-2)^2} + \frac{R_2}{s-j} + \frac{R_3}{s+j}$$

$$\begin{aligned}
R_{11} &= \left[\frac{d}{ds}\left\{\frac{1}{(s^2+1)}\right\}\right]_{s=2} = -\frac{4}{25} \\
R_{12} &= \left[\frac{1}{(s^2+1)}\right]_{s=2} = \frac{1}{5} \\
R_2 &= [(s-j)H(s)]_{s=j} = \left[\frac{1}{(s+j)(s-2)^2}\right]_{s=j} = \frac{1}{8+6j} = \frac{2}{25} - j\frac{3}{50} \\
R_3 &= [(s+j)H(s)]_{s=-j} = \left[\frac{1}{(s-j)(s-2)^2}\right]_{s=-j} = \frac{1}{8+6j} = \frac{2}{25} + j\frac{3}{50} = R_2^* \quad \longrightarrow \quad a = \frac{2}{25} \qquad b = \frac{3}{50}
\end{aligned}$$

$$H(s) = \frac{R_{11}}{s-2} + \frac{R_{12}}{(s-2)^2} + \frac{As+B}{s^2+1}, \quad \text{with} \quad A = \frac{4}{25}, \quad B = -\frac{6}{50}$$

$$h(t) = \left(R_{11}e^{2t} + R_{12}te^{2t} + A\cos t + B\sin t\right)\delta_{-1}(t)$$

Laplace transform (other properties)

Integral property

$$\boxed{\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}\mathcal{L}\left[f(t)\right] = \frac{1}{s}F(s)}$$

direct application:

output response to a step input (with zero initial state), i.e., the **step response**

in t $$y(t) = \int_0^t w(t-\vartheta)\delta_{-1}(\vartheta)d\vartheta = \int_0^t w(t-\vartheta)d\vartheta \overset{\tau = t - \vartheta}{=} \int_0^t w(\tau)d\tau$$

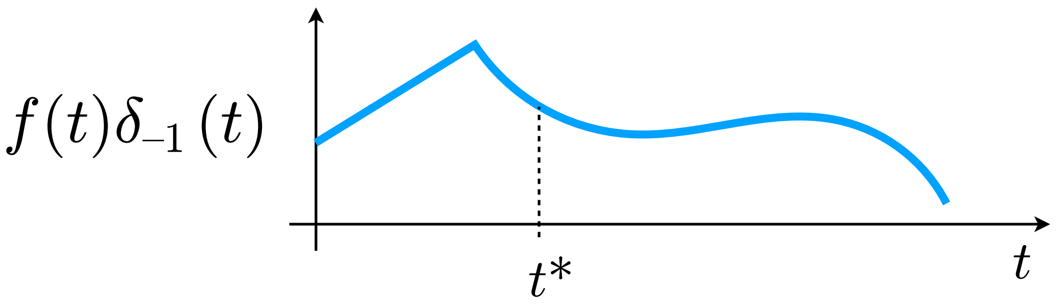
in s $$Y(s) = W(s)U(s) = W(s)\frac{1}{s}$$

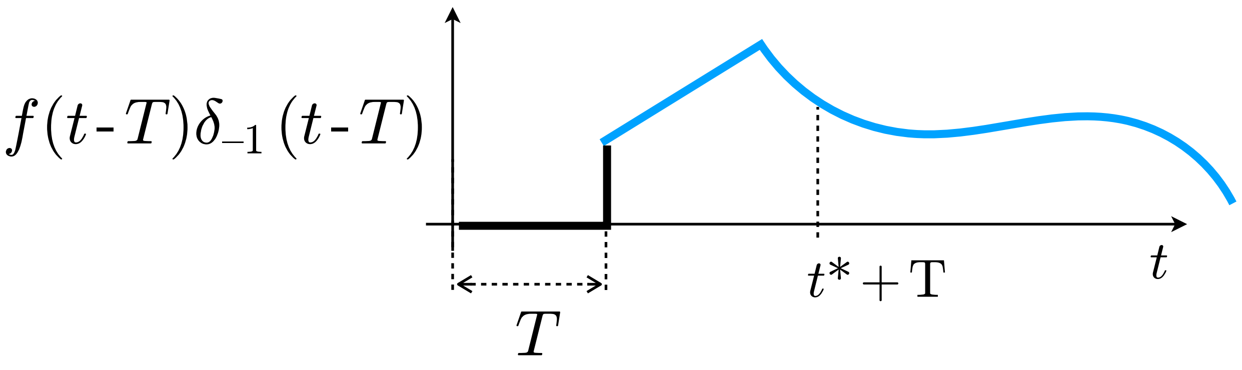
confirmed by integral property

→ the step response is the integral of the impulse response

Laplace transform (other properties)

shifted signal





or, the value of the delayed signal at t_1 is equal to the value that the original signal has at t_1-T

Time shifting property

$$\boxed{\mathcal{L}\left[f(t-T)\delta_{-1}(t-T)\right] = e^{-sT}\mathcal{L}\left[f(t)\delta_{-1}(t)\right] = e^{-sT}F(s)}$$

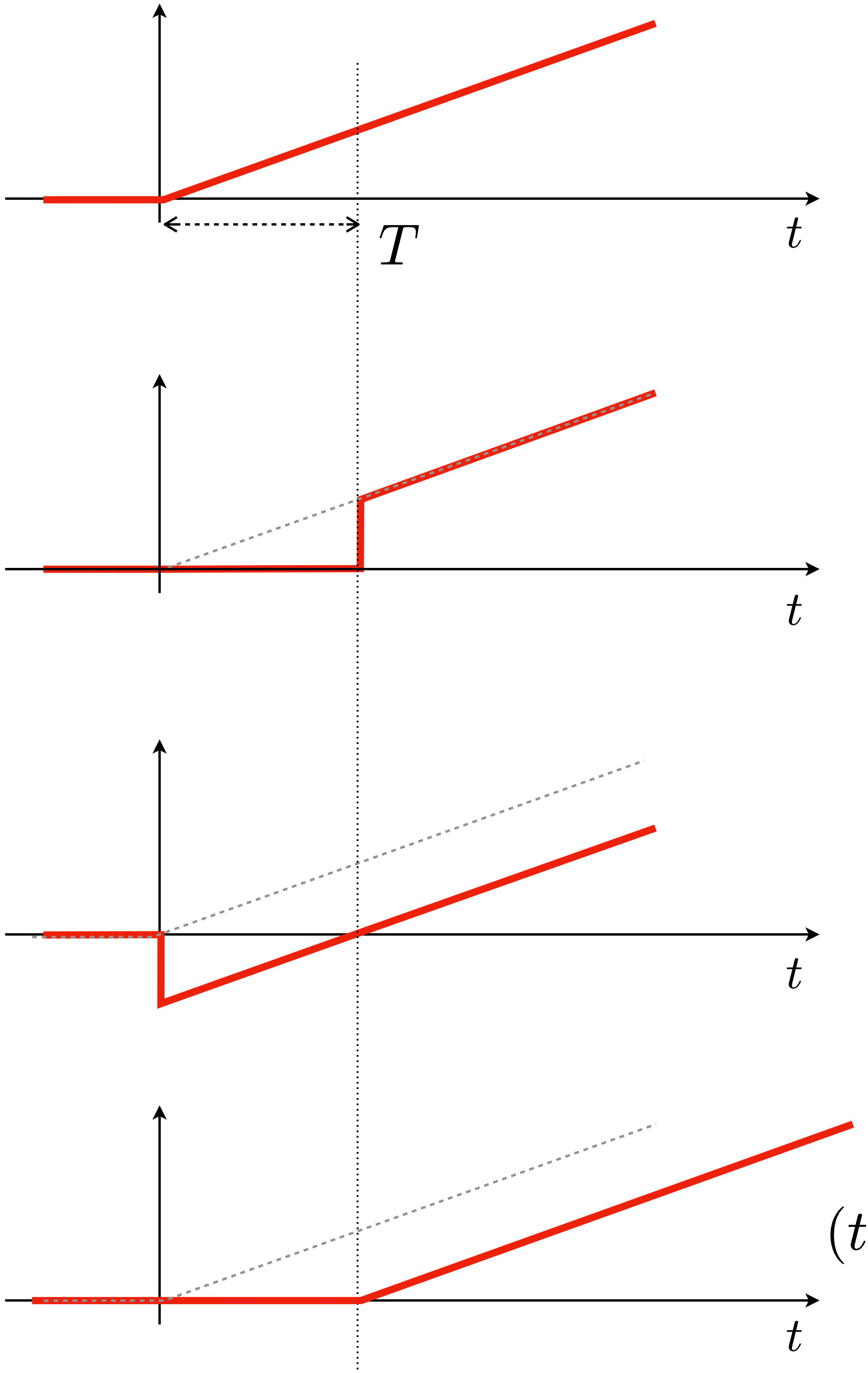
we can use this result in both directions

- if we have a time shifted time function, we find the Laplace transform
- if we have a rational function $F(s)$ multiplied by an exponential in s, we shift the time function resulting from the inverse Laplace transform of $F(s)$

$$e^{-sT}F(s) \longrightarrow f(t-T)\delta_{-1}(t-T)$$

Laplace transform (time shifting)

differences



$t\ \delta_{-1}(t)$

original function to be shifted
by T seconds

$t\ \delta_{-1}(t-T)$

not a shift of $t\ \delta_{-1}(t)$

$(t-T)\delta_{-1}(t)$

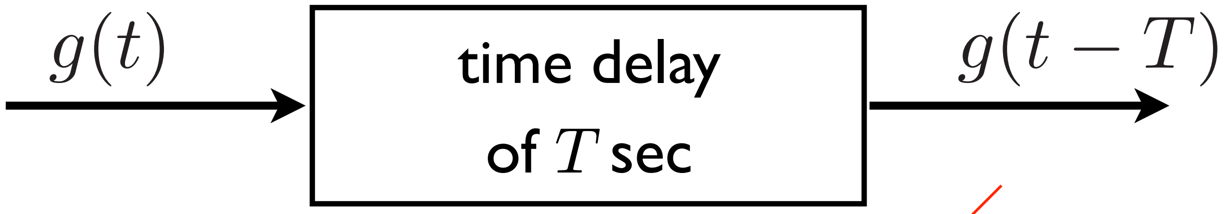
not a shift of $t\ \delta_{-1}(t)$

$(t-T)\delta_{-1}(t-T)$

original function shifted
by T seconds

time delayed signal

assume we have a block that delays a signal by T seconds



$g(t)$

time delay of T sec

$g(t-T)$

we can find its transfer function as the ratio of the Laplace transforms of the output and the input

transfer function of the time delay

$$\frac{\overset{\text{output}}{\mathcal{L}[g(t-T)\delta_{-1}(t-T)]}}{\underset{\text{input}}{\mathcal{L}[g(t)\delta_{-1}(t)]}} = \frac{G(s)e^{-sT}}{G(s)} = e^{-sT}$$

$$\boxed{\text{transfer function of a time delay of } T \text{ sec} = e^{-sT}}$$

not a rational function

Att.: partial fraction expansion is not possible since this transfer function is not rational (it is transcendental)

response to a shifted (delayed) input

assume we know the forced response $Y(s)$ of a system represented by the transfer function $F(s)$ to an input $U(s)$

$$\begin{tikzcd}
u(t) \arrow[r] \arrow[d, "\text{delayed}"] & U(s) \arrow[r] & Y(s) = F(s)U(s) \arrow[r] & y(t) \arrow[ddd, "\text{delayed}"] \\
u(t-T) \arrow[r] & e^{-sT}U(s) \arrow[r] & Y_t(s) = F(s)e^{-sT}U(s) & \\
 & & = F(s)U(s)e^{-sT} & \\
 & & = Y(s)e^{-sT} \arrow[r] & y(t-T)
\end{tikzcd}$$

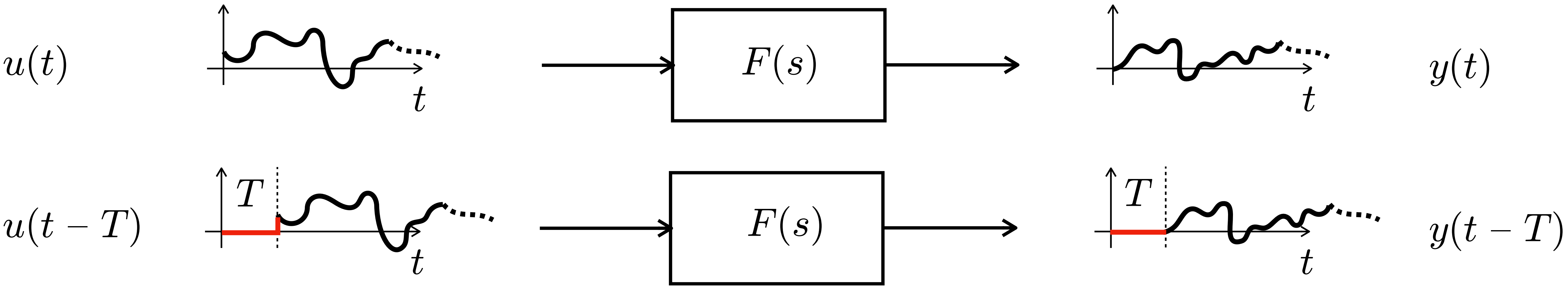
then the forced output response to a shifted by T seconds input $u(t-T)$ is equal to the forced output response $y(t)$ to the unshifted input $u(t)$, shifted by T seconds, that is $y(t-T)$

- the same holds for the state forced response (or state zero-state response)
- when we write $u(t-T)$ we mean $u(t-T)\,\delta_{-1}(t-T)$ (similarly for $y(t-T)$)

response to a shifted (delayed) input

assume we know the forced response $Y(s)$ of a system represented by the transfer function $F(s)$ to an input $U(s)$

$$\begin{tikzcd}
u(t) \arrow[r] \arrow[d, "\text{delayed}"] & U(s) \arrow[r] & Y(s) = F(s)U(s) \arrow[r] & y(t) \arrow[ddd, "\text{delayed}"] \\
u(t-T) \arrow[r] & e^{-sT}U(s) \arrow[r] & Y_t(s) = F(s)e^{-sT}U(s) & \\
 & & = F(s)U(s)e^{-sT} & \\
 & & = Y(s)e^{-sT} \arrow[r] & y(t-T)
\end{tikzcd}$$



to compute the response to $u(t-T)$, if $U(s)$ is a rational function, then we can still use the partial fraction expansion on $Y(s)$ and then translate the resulting time response