Control Systems

Performance

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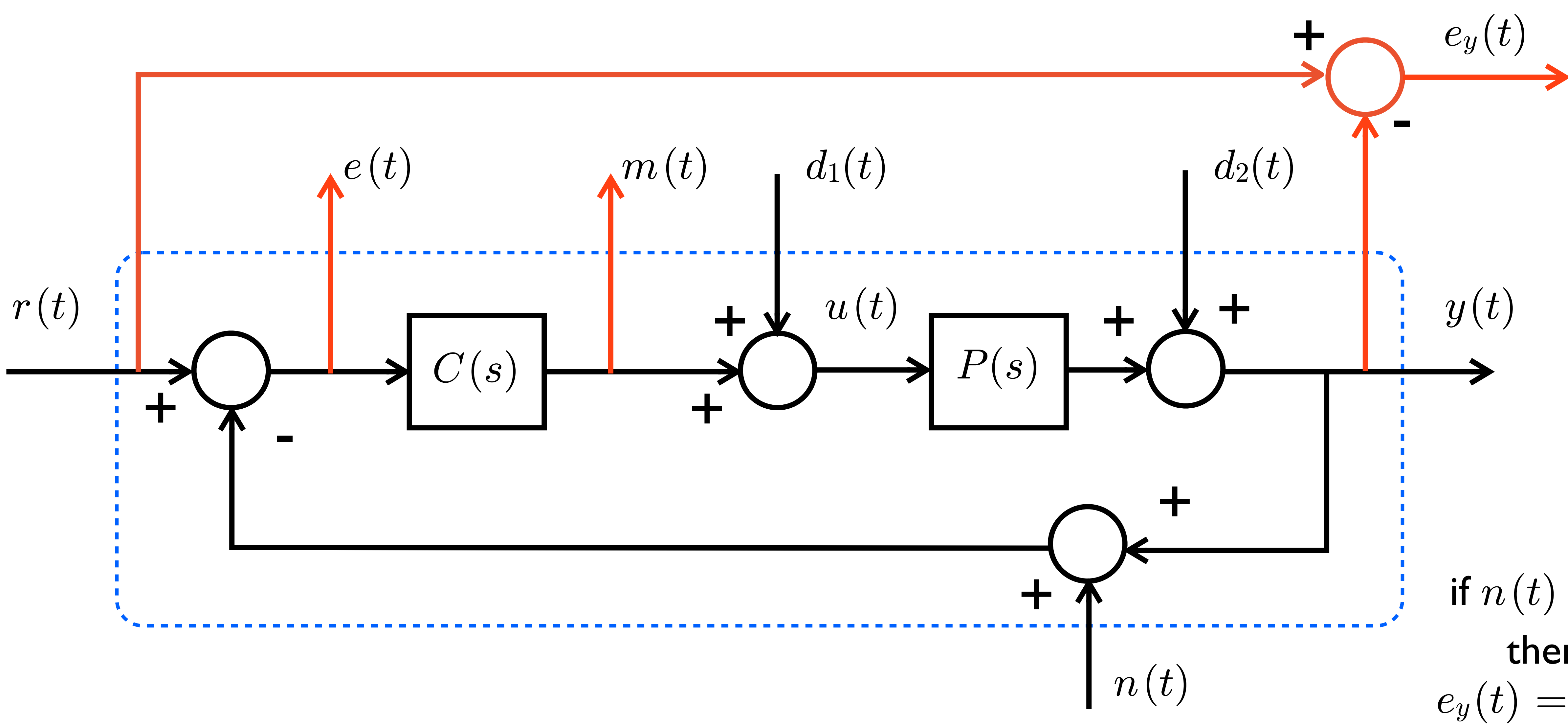
Outline

- loop function approximation
- sensitivity functions approximations
- Parseval theorem
- constraints on the loop function
- the integrator

Feedback control scheme



if $n(t) = 0$
then
$e_y(t) = e(t)$

recall that in the general feedback scheme the outputs of interest are related to the inputs as

$$
\begin{aligned}
y(s) &= T(s)r(s) + P(s)S(s)d_1(s) + S(s)d_2(s) - T(s)n(s) \\
e(s) &= S(s)r(s) - P(s)S(s)d_1(s) - S(s)d_2(s) - S(s)n(s) \\
m(s) &= S_u(s)r(s) - T(s)d_1(s) - S_u(s)d_2(s) - S_u(s)n(s) \\
e_y(s) &= S(s)r(s) - P(s)S(s)d_1(s) - S(s)d_2(s) + T(s)n(s)
\end{aligned}
$$

and being $T(s) = S_u(s)P(s)$

$$
m(s) = S_u(s)(r(s) - P(s)d_1(s) - d_2(s) - n(s))
$$

The closed-loop system is therefore characterized by the three sensitivity functions

$$
\boxed{T(s),\ S(s),\ S_u(s)}
$$

By analyzing the magnitude of their frequency response, we can understand how the closed-loop system behaves w.r.t. sinusoidal inputs $r(t)$, $d(t)$ and $n(t)$

We previously defined

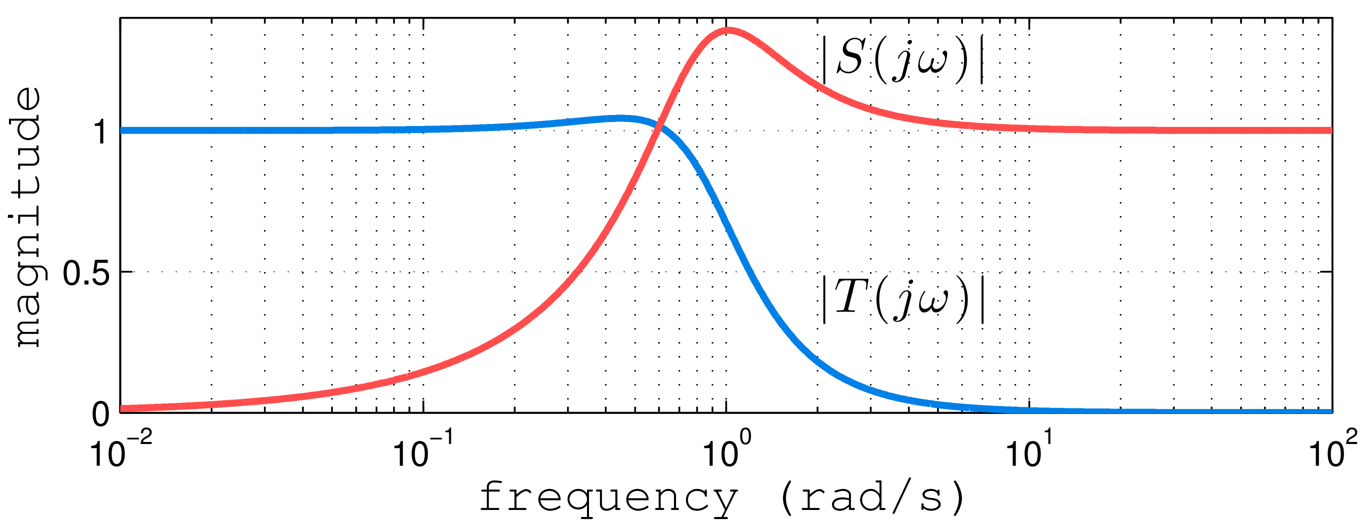
the **loop function** $L(s) = C(s)P(s)$ and

$$S(s) = \frac{1}{1+L(s)}$$ **sensitivity function**

$$T(s) = \frac{L(s)}{1+L(s)}$$ **complementary sensitivity function**

since $S(s) + T(s) = 1$

$$S_u(s) = \frac{C(s)}{1+L(s)}$$ **control sensitivity function**



it doesn't imply that the sum of the magnitudes is equal to 1 (although it is a good approximation at some frequencies)

$$|S(j\omega)| + |T(j\omega)| \neq 1$$

General considerations

$P(s)$ strictly proper

$C(s)$ strictly proper or proper

→ $L(s) = C(s)P(s)$ strictly proper

$$y(s) = T(s)r(s) + P(s)S(s)d_i(s) + S(s)d_o(s) - T(s)n(s)$$

reference

input disturbance

output disturbance

measurement noise

depends also on the filter properties of the plant (discuss)

we can solve the conflicting requirements between $r(t)$ and $n(t)$ by considering signals in different frequency intervals

low $|S(j\omega)|$

high $|T(j\omega)|$

low $|T(j\omega)|$

low frequency

high frequency

ω

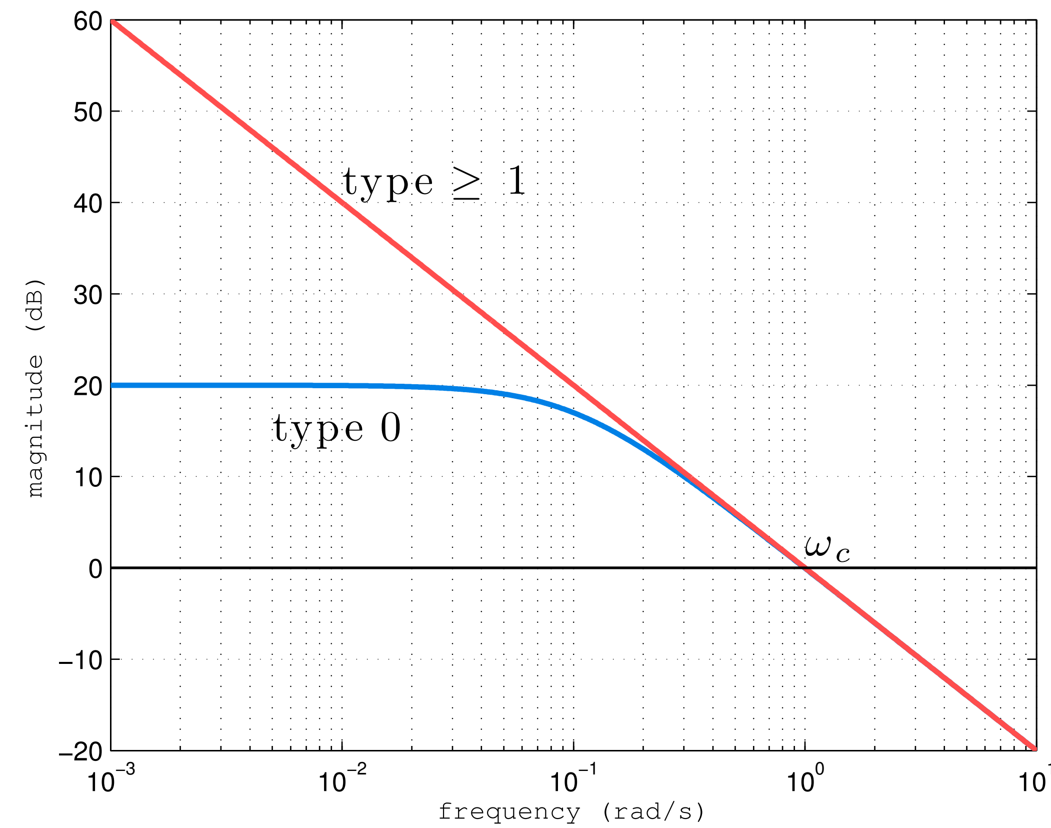
Loop function

either from some static requirements we have poles in $s = 0$ or, for a type 0 system, we require a small value of the error and therefore a high value of the loop gain

↓

at low frequencies the magnitude is usually required to be large

typical behavior of the loop function



approximation

$$|1 + L(j\omega)| \approx \begin{cases} |L(j\omega)| & \text{if} \quad \omega \leq \omega_c \\ 1 & \text{if} \quad \omega > \omega_c \end{cases}$$

bad approximation where $|L(j\omega)|$ close to 1 (i.e. around ω_c)

in dB

$$|1 + L(j\omega)|_{dB} \approx \begin{cases} |L(j\omega)|_{dB} & \text{if} \quad \omega \leq \omega_c \\ 0\,dB & \text{if} \quad \omega > \omega_c \end{cases}$$

Sensitivity function

$$|S(j\omega)| = \frac{1}{|1+L(j\omega)|} \approx |S(j\omega)|^{\text{approx}} = \begin{cases} \frac{1}{|L(j\omega)|} & \text{if} \quad \omega \le \omega_c \\ 1 & \text{if} \quad \omega > \omega_c \end{cases}$$

in dB $\begin{cases} -|L(j\omega)|_{dB} \\ 0\,dB \end{cases}$

the sensitivity function is usually similar to a **high-pass filter**

ok for low frequency reference signals

ok for low frequency disturbance signals

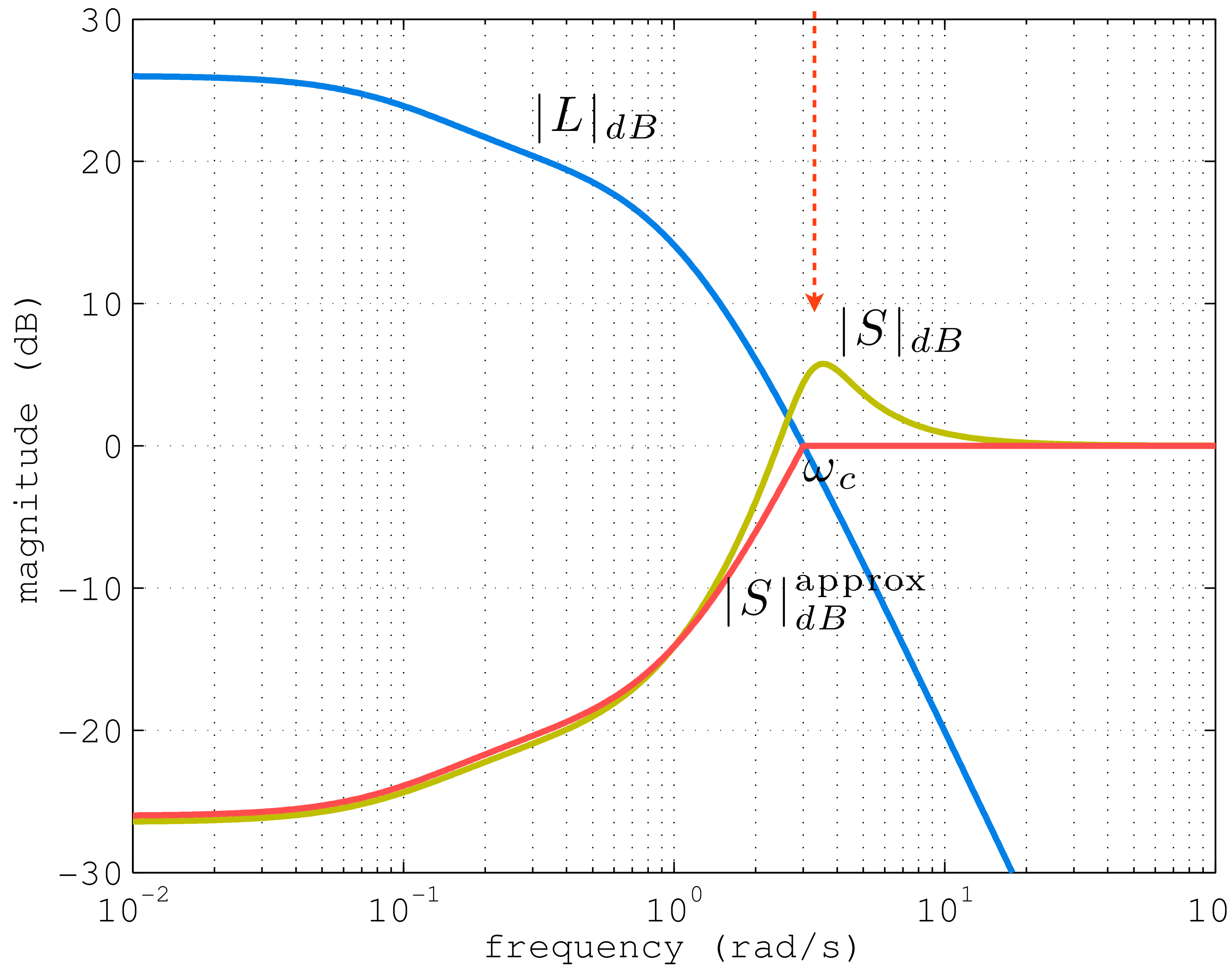
@ low frequency

low sensitivity magnitude

↕

high loop magnitude

bad approximation around ω_c



Complementary sensitivity function

$$|T(j\omega)| = \frac{|L(j\omega)|}{|1+L(j\omega)|} \approx |T(j\omega)|^{\text{approx}} = \begin{cases} 1 & \text{if} \quad \omega \leq \omega_c \\ |L(j\omega)| & \text{if} \quad \omega > \omega_c \end{cases}$$

in dB $\begin{cases} 0\,dB \\ |L(j\omega)|_{dB} \end{cases}$

the complementary sensitivity function is usually similar to a **low-pass filter**

ok for low frequency reference signals

ok for high frequency measurement noise

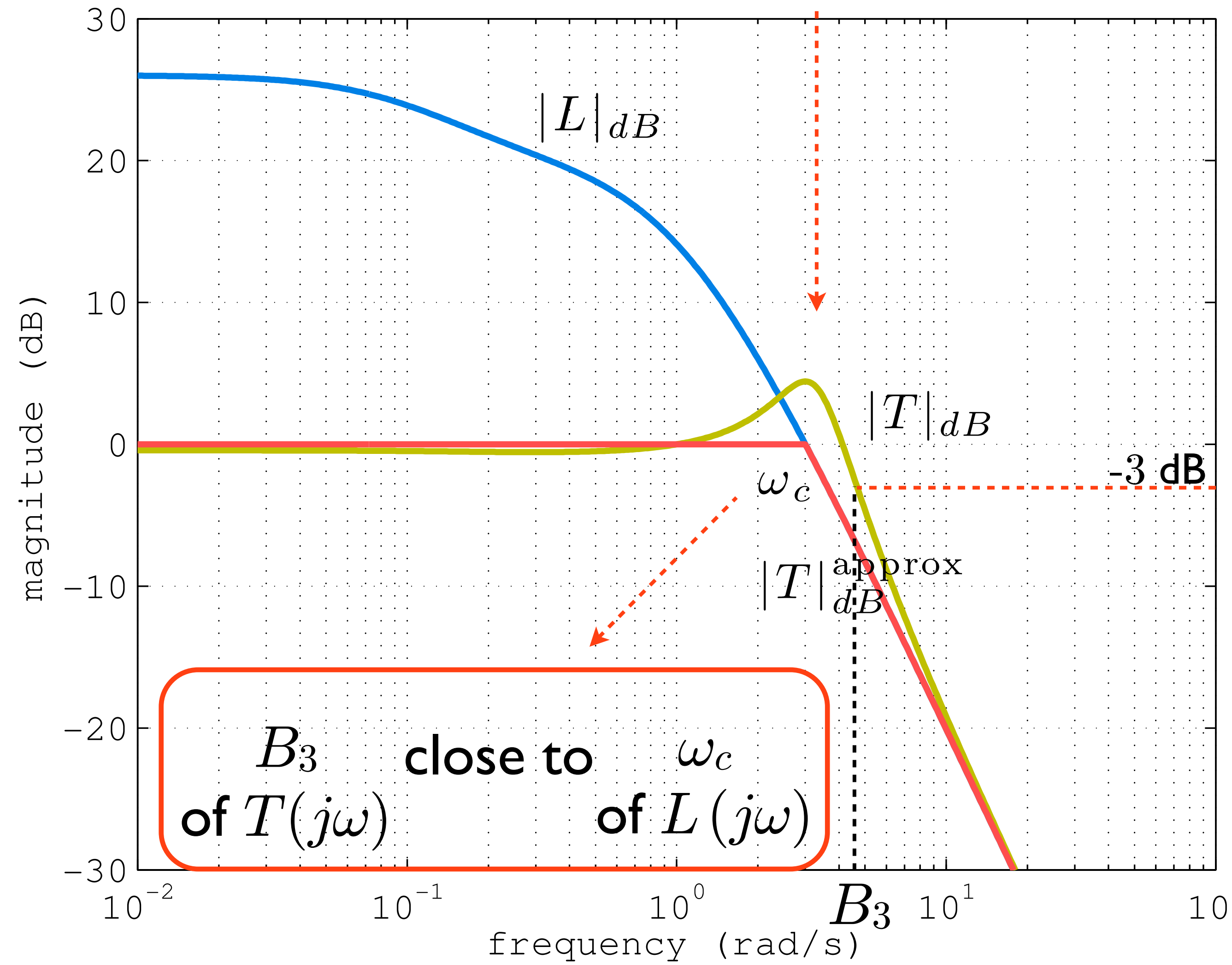
@ high frequency

low complementary sensitivity magnitude

↕

low loop magnitude

bad approximation around ω_c

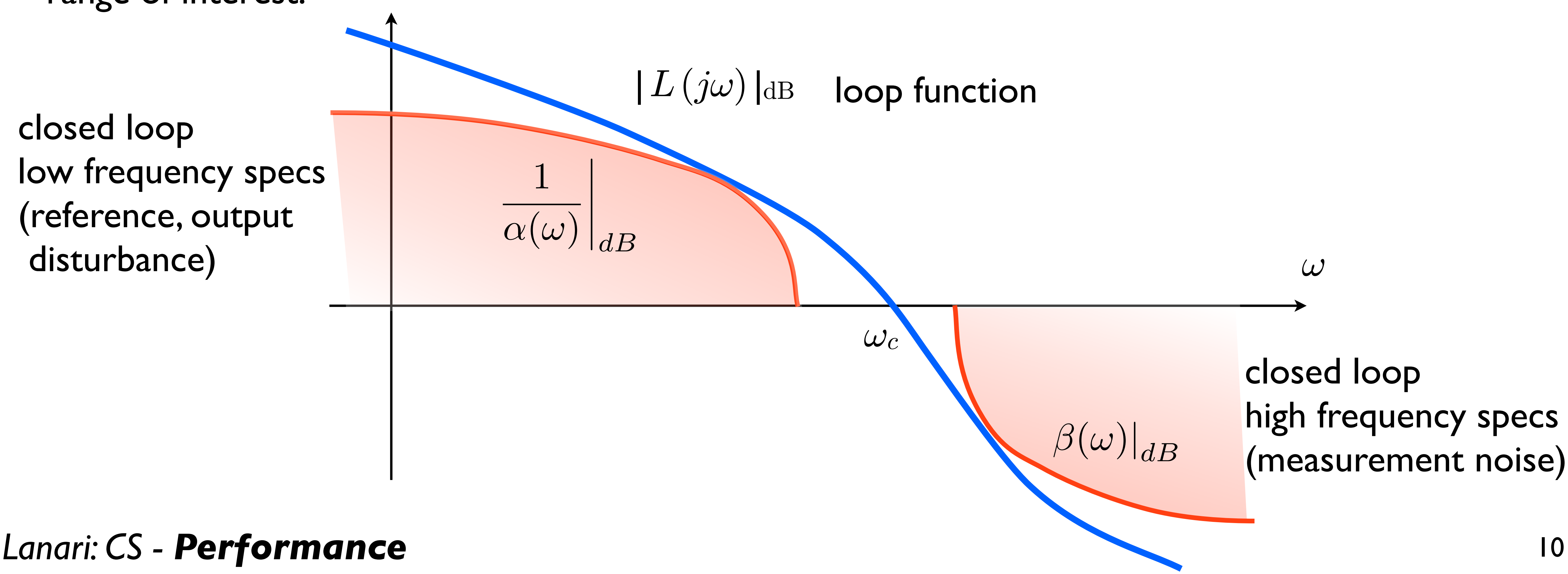


Constraints on the loop function

The previous approximations allow to transform **closed-loop specifications** in open-loop ones

closed-loop specification			open-loop specification (approximated)		
$\lvert S(j\omega)\rvert \leq \alpha(\omega)$	for	$\omega < \omega_c$	$\lvert L(j\omega)\rvert \geq \dfrac{1}{\alpha(\omega)}$	for	$\omega < \omega_c$
$\lvert T(j\omega)\rvert \leq \beta(\omega)$	for	$\omega > \omega_c$	$\lvert L(j\omega)\rvert \leq \beta(\omega)$	for	$\omega > \omega_c$

Since we want attenuation of the disturbances and of the measurement noise and also smaller than one steady state errors w.r.t. sinusoidal references, both α and β are < 1 in the frequency range of interest.



Parseval theorem

important connection between energy in the time domain and the 2-norm in the frequency domain

signal norm with $f(t) = 0$ for $t < 0$

$$\|f(t)\|_2 = \langle f(t), f(t)\rangle^{1/2} = \left(\int_0^\infty f(t)^2 dt\right)^{1/2}$$

square root of energy

$$\|F(s)\|_2 = \langle F(s), F(s)\rangle^{1/2} = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega\right)^{1/2}$$

2-norm

$$\|f(t)\|_2 = \|F(s)\|_2$$

Parseval theorem

applied to the control input $m(t)$

$$\int_0^\infty [m(t)]^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |m(j\omega)|^2\, d\omega$$

(for finite energy signals)

while for sinusoidal signals $m(s) = S_u(s)(r(s) - P(s)d_1(s) - d_2(s) - n(s))$

(other notation) $m(s) = S_u(s)r(s) - S_u(s)P(s)d_i(s) - S_u(s)d_o(s) - S_u(s)n(s)$

one of the two must be "small"

$|r(j\omega)|$

$|P(j\omega)||d_i(j\omega)|$

$|d_o(j\omega)|$

$|n(j\omega)|$

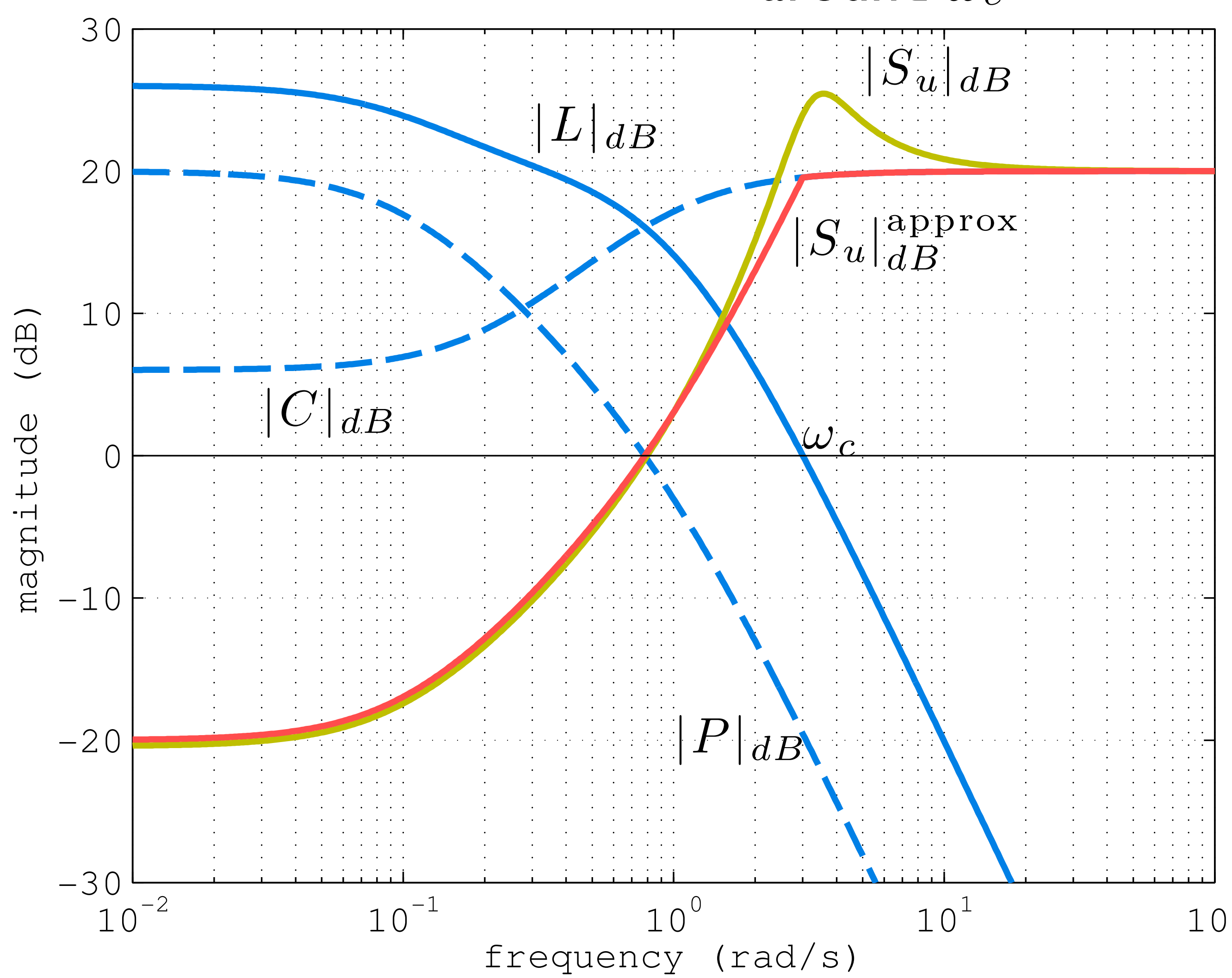
$|S_u(j\omega)| \quad \rightarrow \quad |m(j\omega)|$

Control sensitivity function

$$|S_u(j\omega)| = \frac{|C(j\omega)|}{|1+L(j\omega)|} \approx |S_u(j\omega)|^{\text{approx}} = \begin{cases} \frac{1}{|P(j\omega)|} & \text{if} \quad \omega \leq \omega_c \\ |C(j\omega)| & \text{if} \quad \omega > \omega_c \end{cases}$$

in dB $\begin{cases} -|P(j\omega)|_{dB} \\ |C(j\omega)|_{dB} \end{cases}$

bad approximation around ω_c



at **low frequency**

independent from the controller

depends only on the plant

at **high frequency**

depends only on the controller

and not on the plant

example:

consider the Mass-Spring-Damper system with transfer function from the force to the mass position

$$P(s) = \frac{1}{m\,s^2 + \mu\,s + k}$$

in order to keep the mass at a constant position p_{des}

interpretation of $1/P$ at low frequency, in particular at $\omega = 0$:

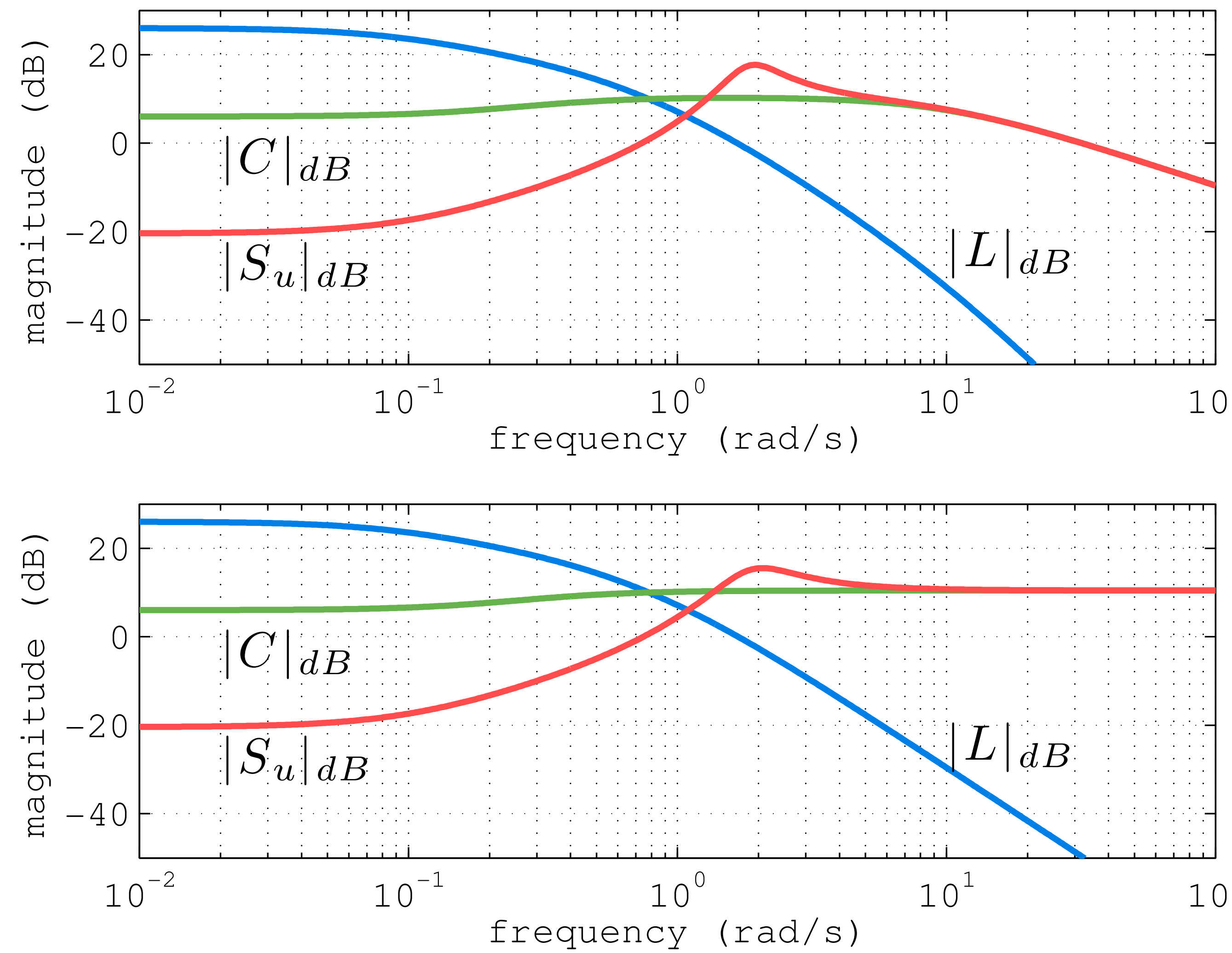
- if I want to keep a mass at 1 and the system gain is 0.5, I need to provide $2 = 1/0.5$ and 0.5 depends only on the plant's characteristics
- the effort depends only from the plant
- It does not mean that the control effort is provided by the plant.

Control sensitivity function

If we consider the other specifications met, it is preferable to have low magnitude for the control sensitivity function and therefore it is better to avoid a proper controller when possible

Techniques that lead to proper controllers:

- PD (approximation)
- pole placement



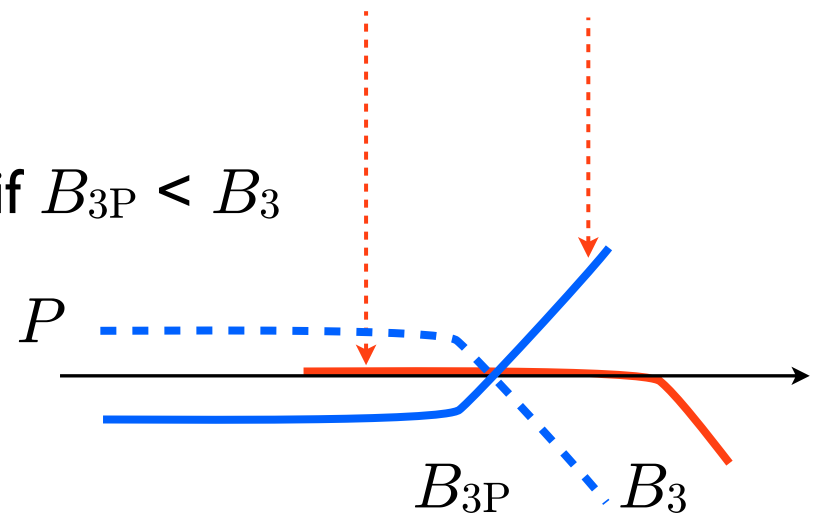
It always depends upon the frequency content of the signals involved

Control sensitivity function

$$S_u(s) = C(s)S(s) = T(s)P(s)^{-1}$$

$$|S_u(j\omega)| = |T(j\omega)||P(j\omega)|^{-1}$$

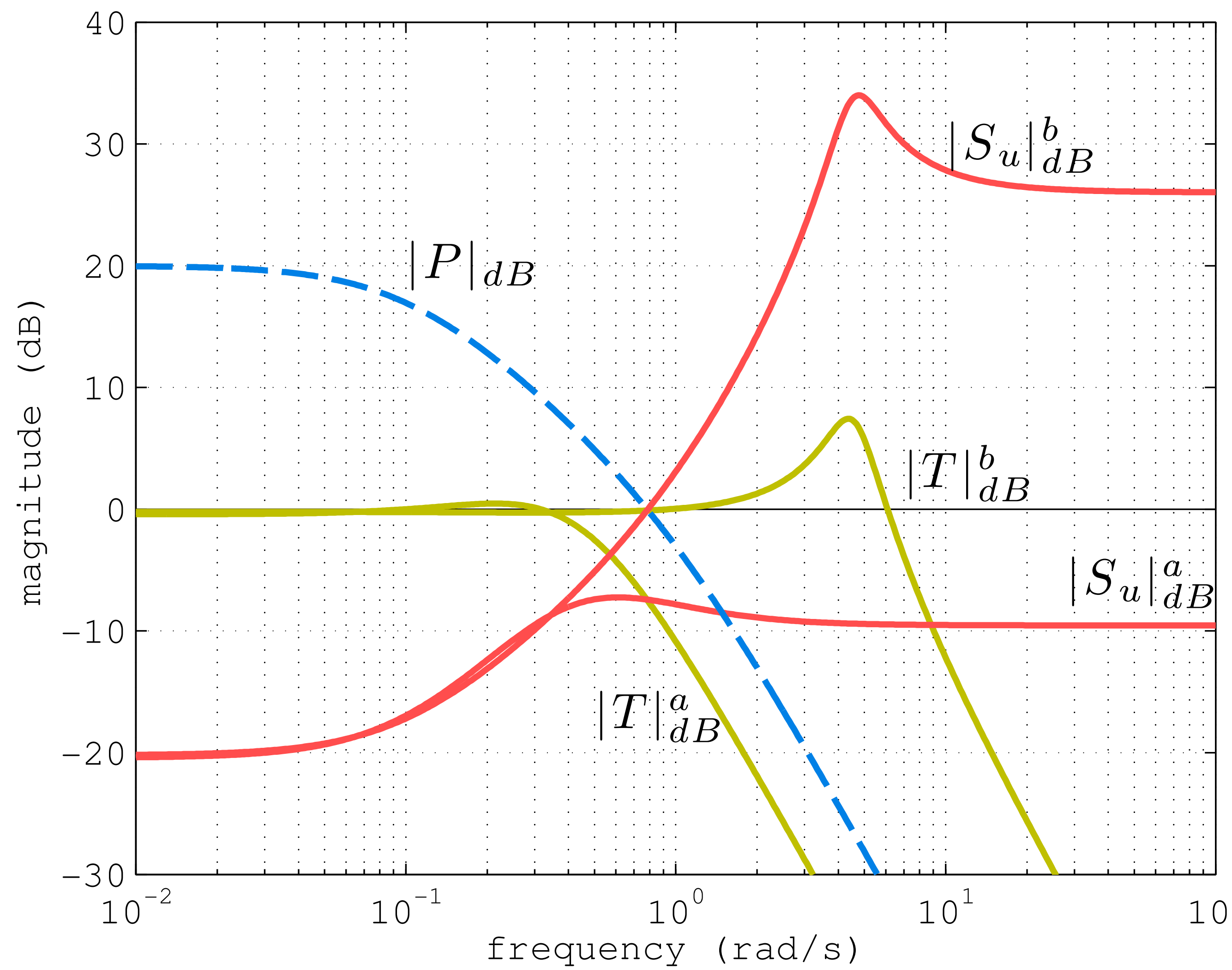
if $B_{3\mathrm{P}}$ < B_3



magnitude increase in the interval $[B_{3\mathrm{P}},\ B_3]$

plant bandwidth $B_{3\mathrm{P}}$

closed-loop bandwidth B_3

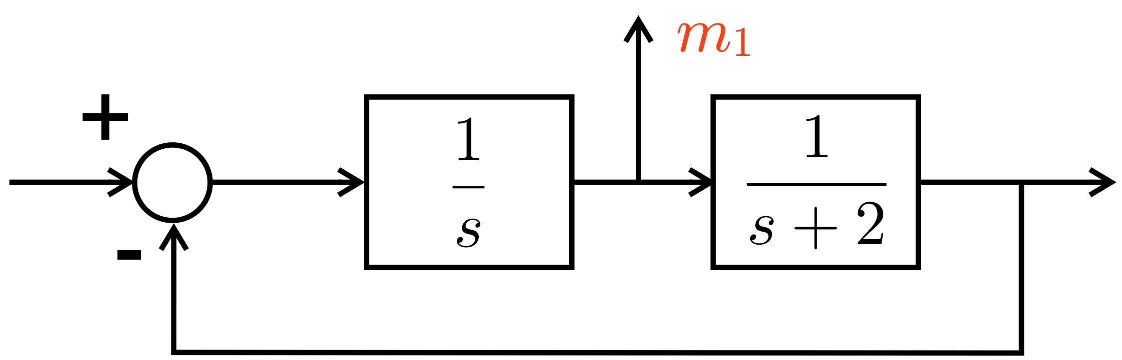


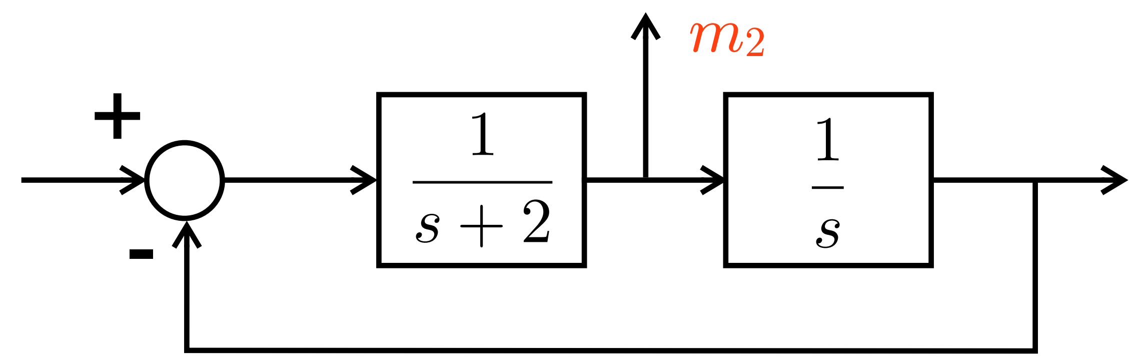
closed-loop bandwidth increase from $B_3{}^a$ to $B_3{}^b$

an increase of the closed-loop bandwidth B_3 w.r.t. the plant's bandwidth $B_{3\mathrm{P}}$ comes at the expense of an increase of the control effort

Control sensitivity function

Effect of an integrator on control effort

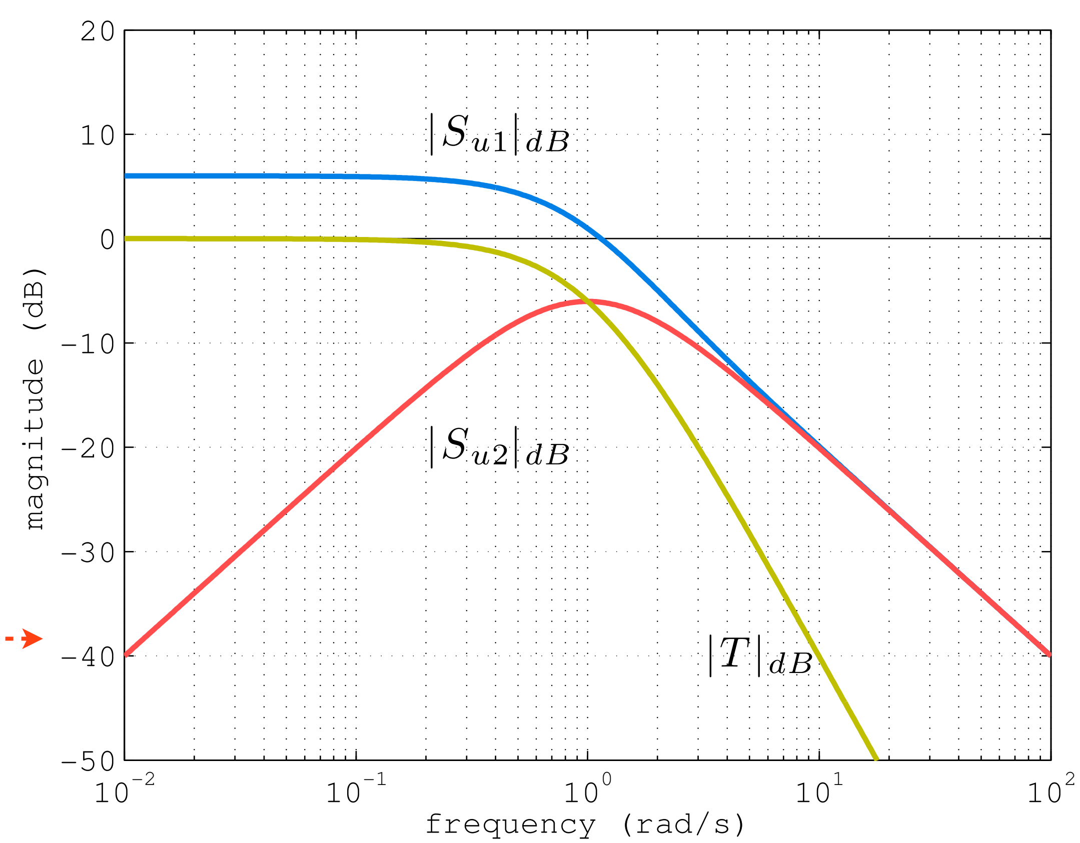


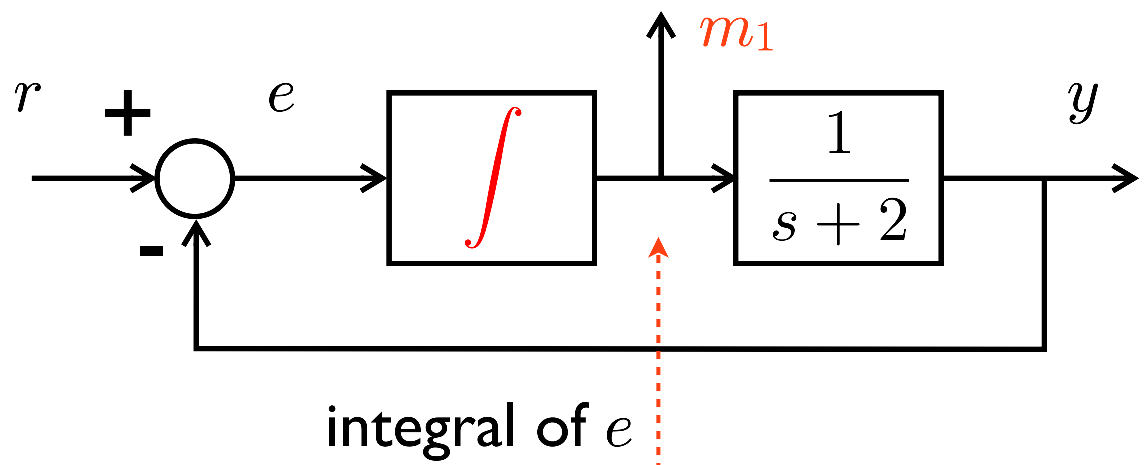


same $L(s)$ then same $T(s)$ and $S(s)$

$$S_{u1}(s) = \frac{s+2}{(s+1)^2} \qquad\qquad S_{u2}(s) = \frac{s}{(s+1)^2}$$

why no control effort at steady-state when applying a constant reference?





at steady-state:

y is equal to the constant r

then the error e is null

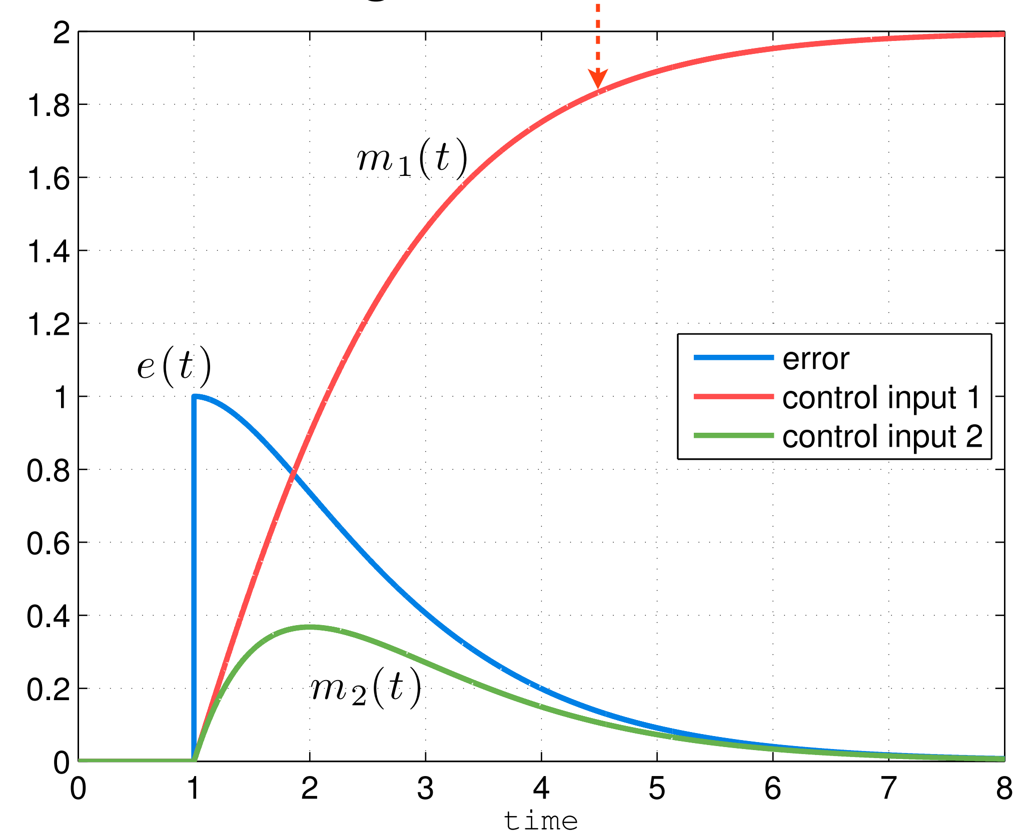
m_1 is the accumulated

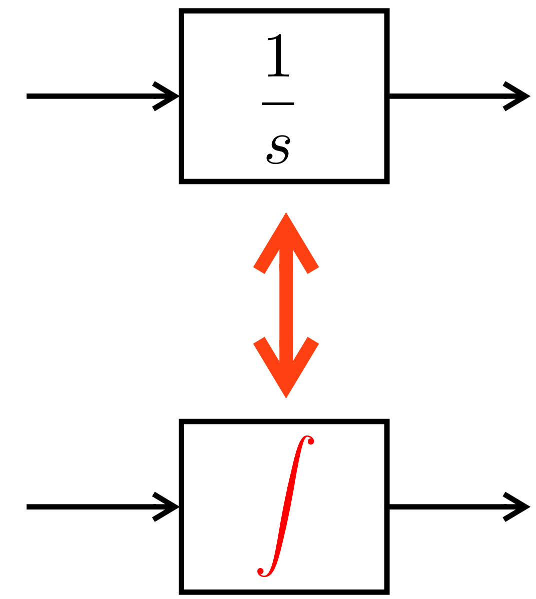
error and drives $1/s+2$

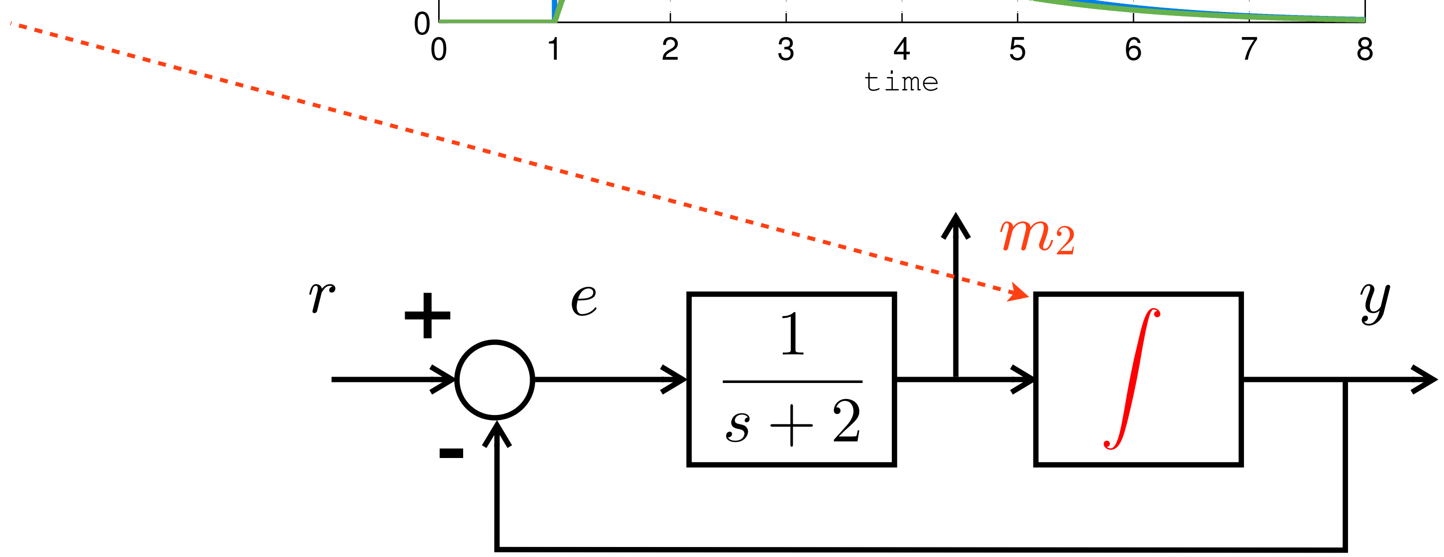
while m_2 is zero but the

output is non-zero due to

the integrator







Vocabulary

English	Italiano
sensitivity function	funzione di sensitività
complementary sensitivity function	funzione di sensitività complementare
control sensitivity function	funzione di sensitività del controllo