Control Systems - January 14, 2025 (with solutions)

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1) Consider the interconnected system $\mathcal{S}$ shown in Fig. 1 with

$$F(s) = \frac{s+1}{s+2}, \qquad G(s) = \frac{1}{s+2}$$

1. Study the controllability and observability of $\mathcal{S}$.
2. Derive the Kalman decomposition of $\mathcal{S}$ with respect to controllability.
3. Draw a block scheme representing the obtained decomposition and explain the resulting transfer function.

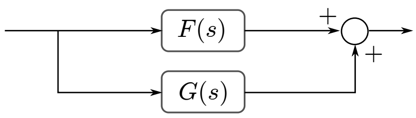

Figure 1: Interconnected system $\mathcal{S}$

Sol. 1) We recognize a parallel interconnection and therefore the overall transfer function is given by

$$W(s) = F(s) + G(s) = \frac{s+1}{s+2} + \frac{1}{s+2} = 1$$

and we immediately understand that both eigenvalues $\lambda_1 = \lambda_2 = -2$ of the interconnected system (recall that the eigenvalues are not modified by this type of interconnection) are hidden. We know that a common eigenvalue in a parallel interconnection becomes both uncontrollable and unobservable due to the interconnection, so in general we would expect the interconnected system to have one pole. We need to investigate further this particular situation since here the resulting transfer function has no poles. As usual, in order to study the loss of controllability and observability (necessary when we note the presence of hidden dynamics) due to the interconnection, we start from the state space realization of each subsystem. Let (x_1, u_1, y_1) and (x_2, u_2, y_2) be respectively the state, input and output of systems $F(s)$ and $G(s)$. Then

$$F(s) = \frac{s+1}{s+2} = 1 - \frac{1}{s+2} \qquad \rightarrow \qquad A_1 = -2, \quad B_1 = 1, \quad C_1 = -1, \quad D_1 = 1$$

$$G(s) = \frac{1}{s+2} \qquad \rightarrow \qquad A_2 = -2, \quad B_2 = 1, \quad C_2 = 1, \quad D_2 = 0$$

and therefore the interconnected system, using the interconnection equations $u = u_1 = u_2$, $y = y_1 + y_2$ and choosing as state

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

is represented by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$

$$y = y_1 + y_2 = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (D_1 + D_2) u = \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + u$$

so the interconnected system is represented by $(A_{//}, B_{//}, C_{//}, D_{//})$

$$A_{//} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \quad B_{//} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad C_{//} = \begin{pmatrix} -1 & 1 \end{pmatrix}, \quad D_{//} = 1$$

We can compute the controllability and observability matrices

$$P = \begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix}, \qquad O = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}$$

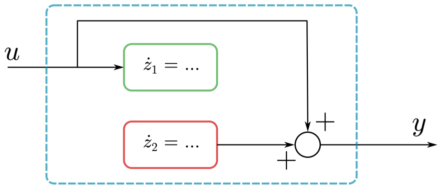

Figure 2: Kalman decomposition of $\mathcal{S}$ w.r.t. controllability

which are both singular and have rank equal to 1. So there is a one-dimensional uncontrollable system and a one-dimensional unobservable subsystem. We can perform the Kalman decomposition w.r.t. controllability or observability.

- *Kalman decomposition w.r.t. controllability.*

We can choose the new state $z = Tx$ with

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = Tx \qquad T^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \Longrightarrow \quad T = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

and therefore the new system representation is

$$A_c = TA_{//}T^{-1} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \quad B_c = TB_{//} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C_c = C_{//}T^{-1} = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad D_c = D_{//} = 1$$

The dynamic equations in the new coordinates are

$$\begin{aligned} \dot z_1 &= -2z_1 + u \\ \dot z_2 &= -2z_2 \\ y &= z_2 + u \end{aligned}$$

with the block scheme shown in Fig. 2.

From the decomposition shown in Fig. 2 we understand that the controllable subsystem is unobservable while the uncontrollable subsystem is observable and therefore there is no eigenvalue which is simultaneously controllable and observable. The resulting transfer function has therefore no poles but the presence of a non-zero feedthrough term $D = 1$ gives the final transfer function $W(s) = 1$.

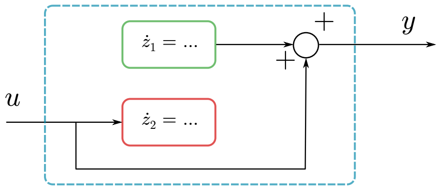

Figure 3: Kalman decomposition of $\mathcal{S}$ w.r.t. observability

- *Kalman decomposition w.r.t. observability.*

We can choose the new state $z = Tx$ with

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = Tx \qquad T^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \Longrightarrow \quad T = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

and therefore the new system representation is

$$A_o = TA_{//}T^{-1} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \quad B_o = TB_{//} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C_o = C_{//}T^{-1} = \begin{pmatrix} -1 & 0 \end{pmatrix}, \quad D_o = D_{//} = 1$$

The dynamic equations in the new coordinates are

$$\begin{aligned} \dot z_1 &= -2z_1 \\ \dot z_2 &= -2z_2 + u \\ y &= -z_1 + u \end{aligned}$$

with the block scheme shown in Fig. 3

Similar considerations with respect to the previous decomposition hold. A different change of coordinates does not alter the observable or controllable dynamics.

Some common errors

- Once the state space representation has been found, if it turns out that the eigenvalues of the interconnected dynamic matrix are different form the eigenvalues of the single subsystems it means there has been an error in the derivation; we know that the eigenvalues in a series interconnection are the union of the eigenvalues of the single subsystems.
- It may appear strange, but some (and not just a few) did an error in computing the sum of the transfer functions $F(s) + G(s)$ ...
- Some have obtained a 2-dimensional state space ($n = 2$, $x \in \mathbb{R}^2$) representation for each 1-dimensional system $F(s)$ and $G(s)$.
- Some have not considered (or did not do it correctly) the feedthrough term D_1 in the realization of the proper transfer function $F(s)$.
- Errors in computing the kernel of the observability matrix or the image of the controllability matrix are unfortunately common

2) Consider the feedback system having the following open loop system

$$F(s) = K\frac{(s+1)^2}{(s^2+1)^2}, \qquad K \in I\!R$$

1. Study the closed loop stability with the Routh criterium as a function of K.
2. Draw the positive and negative corresponding root locuses and verify the previous result.
3. Verify the closed loop stability with the Nyquist plot.

Sol. 2) The closed loop pole polynomial is

$$p(s,K) = (s^2+1)^2 + K(s+1)^2 = s^4 + (K+2)s^2 + 2K\,s + K + 1$$

Since there is a missing coefficient (of the term s^3) the necessary condition for having all roots with negative real part is not verified and we do not build the Routh table (which would have a zero element in the first column). We can directly conclude that the closed loop system will never be asymptotically stable independently from the values of the gain K.

In order to draw the correct root locus we first note that the zero in $s=-1$ and the poles in $s=\pm j$ all have multiplicity 2 so these are also singular points with multiplicity $\mu=2$ of the root locus with alternating positive and negative branches at 90°, entering and exiting. What is less clear is the orientation of the four branches in $\pm j$; this orientation may affect the crossing of the imaginary axis. Being $n-m=2$ the center of asymptotes is in $s_o=1$ so clearly the closed loop system will not be asymptotically stable for high positive values of the gain.

Some insight can be obtained from the computation of the singular points using the simplified formula

$$\frac{2}{s+j}+\frac{2}{s-j}-\frac{2}{s+1} = 2\left(\frac{1}{s+j}+\frac{1}{s-j}-\frac{1}{s+1}\right) = 2\left(\frac{2s}{s^2+1}-\frac{1}{s+1}\right) = 0 \qquad \Leftrightarrow \qquad s^2+2s-1=0$$

Being the two solutions $s^* = -1 \pm \sqrt{2}$, that is $s_1^* \approx -2.41$ and $s_2^* \approx 0.41$ real, these are singular points. The resulting root locus (positive and negative) is shown in Fig. 4.

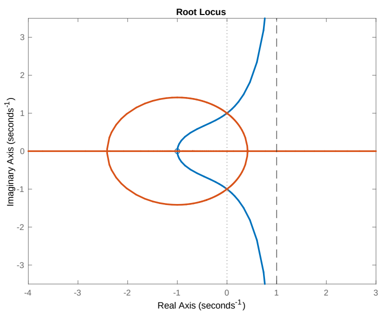

Figure 4: Root locus with the vertical asymptotes and center of asymptotes in $s_o=1$, two open loop poles in $\pm j$ and one zero in -1, all with multiplicity 2.

Note that without computing explicitly the singular point or without considering the stability previously found result, we could have had two possible locations of the rightmost singular point as shown in Fig. 5. However in the plot on the right we would have obtained an interval of negative values for K for which the closed loop would have been asymptotically stable. Therefore we could have directly stated that the singular point would be positive in order to be consistent with the previous stability result (necessary condition on the pole polynomial).

The corresponding Bode and Nyquist plots for $K=1$ are shown in Fig. 6. Note that the purely imaginary poles have multiplicity two so the phase shift due to the trinomial term at the natural frequency $\omega_n = 1$ rad/s is -2π. The Nyquist plot for negative values of K is obtained rotating the one shown by $-\pi$ (the phase shift introduced by the negative gain). In both cases it is clear that the number of counter-clockwise encirclements is negative (-2 or -1 when K is negative for $K<-1$) and thus the closed loop system is never asymptotically stable (as already stated).

Some common errors

- The necessary condition for all the roots of a polynomial to have negative (strictly) real part is that all the polynomial coefficients should have the same sign. A missing coefficient (here in the pole polynomial) is a different sign however this means only that the necessary condition is not satisfied and not that the

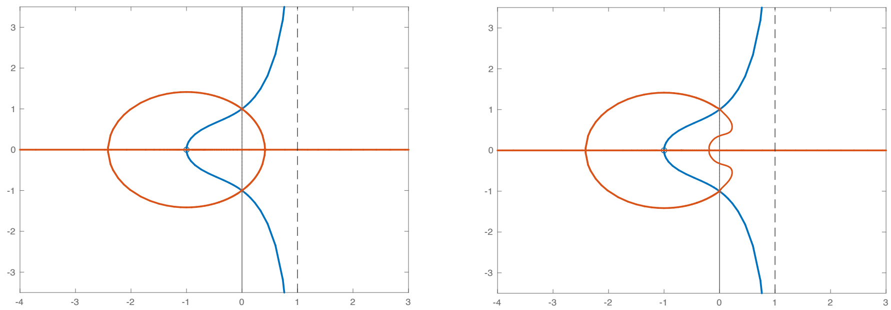

Figure 5: Two possible root locus plots with different rightmost singular point.

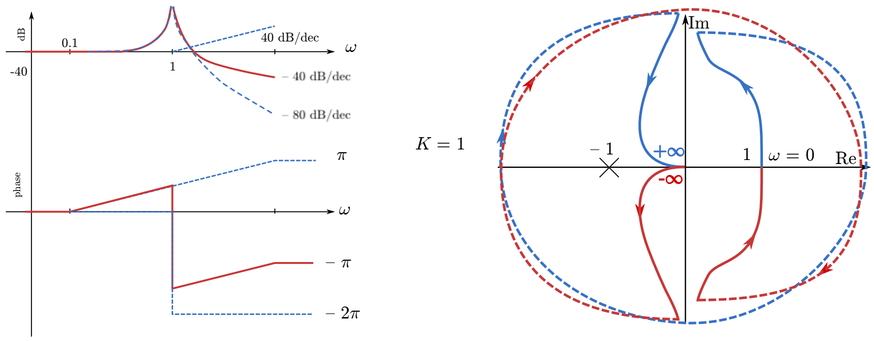

Figure 6: Bode and Nyquist plots.

system is unstable; the system could have poles with real part equal to 0. For example the following pole polynomials correspond to marginally stable systems

$$s^3 + s = s(s^2+1) \qquad \longrightarrow \qquad \text{roots (poles)}: \quad -1, \pm j$$
$$s^4 + 10s^2 + 9 = (s^2+1)(s^2+9) \qquad \longrightarrow \qquad \text{roots (poles)}: \quad \pm j, \pm 3j$$

- There was no requirement of finding a controller which stabilizes the system in closed loop; always read the text carefully.
- Drawing the closure at infinity with a tiny almost invisible line because you are insecure does not help ...

3) Consider the following dynamic model of the car shown in Fig. 7

$$m\,\ddot{p}(t) + \mu\,\dot{p}(t) = f(t)$$

with m the car total mass, μ a friction coefficient, $f(t)$ the traction force and $p(t)$ the distance of the car's center of mass from a wall. Moreover d denotes the length from the car's center of mass to the rear.

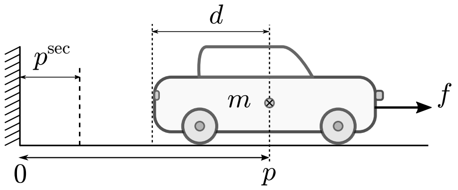

Figure 7: Interconnected system $\mathcal{S}$

We want to design an automatic parking system which brings the rear of the car at a security distance p^{sec}.

1. Draw the corresponding control scheme with the signals of interest.
2. Having $m = 1000$ kg and $\mu = 100$ N$\cdot$ s/m, determine the static controller $C_1(s) = K_c$ such that we have $\omega_c^* = 0.1$ rad/s and a phase margin of at least 45°.
3. Determine a different (not necessarily static) controller $C_2(s)$ such that the car, during the transient, has a faster and better transient behavior with respect to $C_1(s)$.
4. Explain to a driver, what do the specifications on ω_c^* and the phase margin mean in practical terms for the car.

Sol. 3) In terms of control, we want the rear of the car to asymptotically reach the p^{sec} location or, equivalently, we want p to reach $p^{\text{sec}} + d$. Assuming we can measure the position p, we obtain the feedback control scheme of Fig. 8. The car's transfer function can be obtained from the given dynamic model choosing, for example, p and $\dot{p}$ as state components and thus obtain the following state space representation matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -\mu/m \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1/m \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad D = 0$$

Note that we could have chosen as state vector

$$w = \begin{pmatrix} p - d \\ \dot{p} \end{pmatrix}$$

leading to the same state space representation and thus same transfer function (since there is no term in p in the differential equation) but a different control scheme in which the reference would have been simply p^{sec}. In this case, however, we should state explicitly that we measure $p - d$ and not p.

The resulting transfer function is

$$P(s) = \frac{1}{m\,\ddot{s} + \mu\,s} = \frac{1}{s(m\,s + \mu)}$$

Equivalently we could have taken the Laplace transform of the differential equation assuming zero initial state.

We note the presence of a pole in $s = 0$ in the transfer function which is equivalent to an integrator which clearly relates the velocity $\dot{p}$ to the position p. This integrator (or pole in $s = 0$) makes the control system of at least type 1 and therefore, since we want to track a constant reference $p^{\text{sec}} + d$, we do not need to introduce any further poles in $s = 0$ in the controller.

Note also that with a positive static controller $C_1(s) = K_c$ we can stabilize the closed loop system since the corresponding pole polynomial becomes

$$p(s, K_c) = m\,s^2 + \mu\,s + K_c$$

This static controller has the effect of a spring.

We know that it is sufficient to guarantee a zero steady state error for $p(t)$ at the desired location $p^{\text{sec}} + d$, however the way the car reaches this steady state is also important. In particular, an excessive overshoot would let the rear of the car bump into the wall. With the given values of m and μ we can draw the Bode plots of the open loop system (see Fig. 9).

$$L_1(s) = C_1(s)P(s) = \frac{K_c}{s(m\,s + \mu)} = \frac{K_c}{100\,s(10\,s + 1)}$$

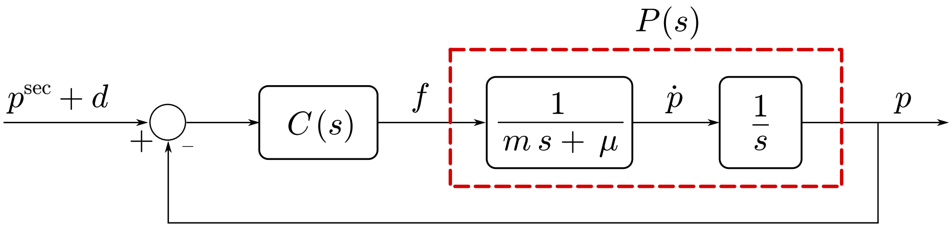

Figure 8: Control scheme - automatic parking.

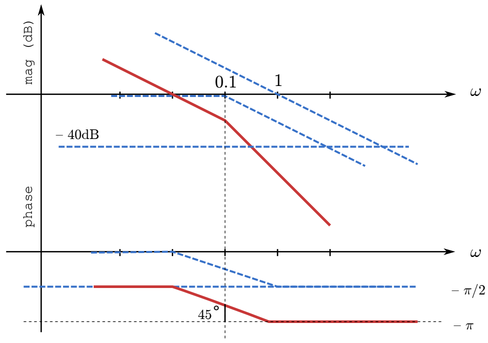

Figure 9: Control scheme - Bode plots.

In order to achieve the required crossover frequency and phase margin, it is sufficient to choose the gain $K_c = 10$ (or $K_c = 10^{23/20}$ if we consider the correct value of the magnitude at 0.1 rad/s).

A better transient should be characterized by a higher crossover frequency and, possibly, also a higher phase margin. This could be achieved through a correct choice of a lead function and an extra gain; as a numerical example one could obtain a crossover frequency of 0.3 rad/s with a phase margin greater than 45° with $m_a = 14$ and $\omega\,\tau = 1.4$ and a gain to obtain the desired crossover frequency. For computations done by hand approximate values are clearly accepted. Instead, as a numerical example, we choose

$$C_2(s) = 10^{34.4/20}\frac{1 + 4.66s}{1 + 4.66/14s}$$

and we achieve $\omega_c = 0.29$ rad/s and $PM = 67°$.

These are open loop requirements which originate from closed loop ones on the bandwidth and resonance peak. In the time domain it means we are requiring a faster convergence to the desired final position (higher closed loop bandwidth) with less overshoot (in order to avoid any bumping of the car with the wall). A comparison of the two controllers is performed as an example: in Fig. 10 the position of the rear of the car is shown starting from a distance of 1.5 m with zero velocity and $p^{\text{sec}} = 0.5$ m.

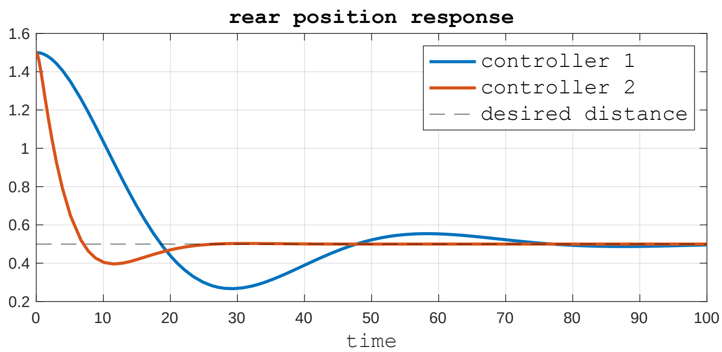

Figure 10: Control scheme comparison - rear position starting from a distance of 1.5 m and $p^{\text{sec}} = 0.5$ m.

Some common errors

- Choosing the state as just the velocity $x = \dot{p}$ could be possible if looking only at the dynamic equation which can be seen as a first order differential equation in the velocity

$$\frac{d}{dt}\dot{p} = -\frac{\mu}{m}\dot{p} + \frac{1}{m}f = \dot{x} = -\frac{\mu}{m}x + \frac{1}{m}f$$

but since we want to regulate the position, we are going to measure the position and do a feedback from this measure; therefore to express the position as a linear combination of the state $y = Cx$, we need to include also the position in the state.

- Moreover, choosing the velocity as output but the correct state (position and velocity) would lead to a not asymptotically stable hidden dynamics which cannot be modified by the output feedback.
- You should not interpret d as a disturbance, it only defines the correct reference signal if we measure the position p. Some have put d as an output disturbance; this implies that the observed and measured variable is $p + d$ and not p.
- Some have written the transfer function $P(s)$ in a wrong Bode canonical form.

4) Compute the output steady state response of the system

$$P(s) = \frac{20}{s+10}$$

to the input $u(t) = 3(t-1)\delta_{-1}(t-1) - \sqrt{2}\sin(10\,t)\delta_{-1}(t)$

Sol. 4) First note that the system is asymptotically stable so the steady state exists. We rewrite the transfer function in its Bode canonical form

$$P(s) = \frac{20}{s+10} = \frac{2}{1+s/10}$$

Then, being the system linear, the steady state response of a linear combination of signals is the same linear combination of the single steady state responses to the single signals.

We first compute the steady state response of the translated ramp. Since the forced response to a translated signal coincides with the translation of the response to the untranslated signal, we first compute the steady state response to the untranslated ramp $t\delta_{-1}(t)$. The forced response is

$$u_1(t) = t\delta_{-1}(t) \quad \longrightarrow \quad y_1(s) = P(s)\frac{1}{s^2} = \frac{20}{s^2(s+10)} = \frac{0.2}{s+10} - \frac{0.2}{s} + \frac{2}{s^2}$$

and therefore the steady state response is

$$y_{1,ss}(s) = -\frac{0.2}{s} + \frac{2}{s^2} \quad \longrightarrow \quad y_{1,ss}^{\text{temp}}(t) = (-0.2 + 2\,t)\delta_{-1}(t)$$

which, when translated, gives

$$y_{1,ss}(t) = (-0.2 + 2\,(t-1))\delta_{-1}(t-1).$$

Note that at steady state, the response to a step input or its translation in time coincide, moreover the coefficient of the leading term coincides with the system gain. Here, however, we have a ramp and this does not hold anymore.

For the sinusoidal signal, we know that in general

$$u(t) = \sin(\bar{\omega}\,t)\delta_{-1}(t) \quad \longrightarrow \quad y_{ss}(t) = |P(j\bar{\omega})|\sin(\bar{\omega}\,t + \angle P(j\bar{\omega}))\delta_{-1}(t)$$

Since in this case $\bar{\omega}$ is exactly the cut-off frequency of the binomial term, we know its contribution in terms of magnitude and phase that is -3 dB (or $1/\sqrt{2}$) and $-45°$. So we have

$$u_2(t) = -\sqrt{2}\sin(10\,t)\delta_{-1}(t) \quad \Longrightarrow \quad y_{2,ss}(t) = -\sqrt{2}\frac{2}{\sqrt{2}}\sin(10\,t - \frac{\pi}{4})\delta_{-1}(t)$$

(where 2 is the system gain, $-\sqrt{2}$ the amplitude of the input signal and $1/\sqrt{2}$ is the magnitude of the binomial term at the given frequency) so that the overall steady state response is

$$y_{ss}(t) = 3(-0.2 + 2\,(t-1))\delta_{-1}(t-1) - 2\sin(10\,t - \frac{\pi}{4})\delta_{-1}(t)$$

For illustration, Figure 11 shows the forced response of $P(s)$ to the given input and the steady state response.

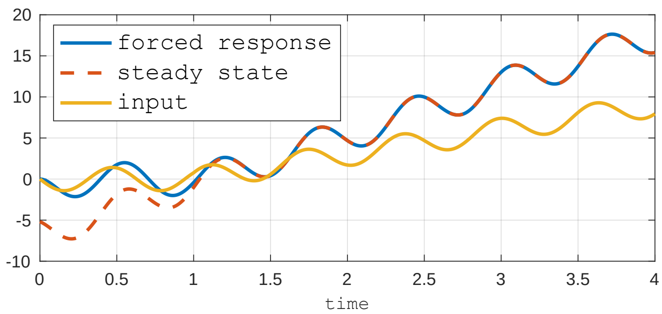

Figure 11: Forced and steady state response.

Some common errors

- Wrong application of the final value theorem: both components of the input (ramp and sinusoidal) give rise to terms in the forced response which do not admit the application of the Final Value Theorem due to the presence of roots (in this example) with zero real part at the denominator of $Y(s)$. When we omit the time shift we have

$$\lim_{s\to 0} sY(s) = \lim_{s\to 0} s\frac{20}{s+10}\left(3e^{-s}\frac{1}{s^2} - \sqrt{2}\frac{10}{s^2+100}\right)$$

- Many computed the forced response to the ramp input but did not eliminate the transient part in the steady state.
- In the computation of the steady state response to a sinusoidal input, many tried to go through the expansion in residues (which is possible) but we know that the frequency response has all the information required (magnitude and phase) for computing directly the steady state response to a sinusoidal input.
- Remember that we consider signals that are equal to 0 for negative time; this is why we write for example $\sin(\omega\, t)\delta_{-1}(t)$ which has unilateral Laplace transform

  $$\mathcal{L}\left[\sin(\omega\, t)\delta_{-1}(t)\right] = \frac{\omega}{s^2 + \omega^2}$$

  Moreover the Laplace transform of the product of two time functions is not the product of the Laplace transforms of each time function, that is

  $$\mathcal{L}\left[f(t)g(t)\right] \neq \mathcal{L}\left[f(t)\right]\mathcal{L}\left[g(t)\right]$$