Autonomous and Mobile Robotics
Solution of Midterm Class Test, 2024/2025

Solution of Problem 1

(a) The kinematic constraints acting on the robot are the following (one pure rolling condition per wheel):

$$\begin{aligned} \dot{x}\sin\theta - \dot{y}\cos\theta &= 0 \\ \dot{x}_f\sin(\theta+\phi) - \dot{y}_f\cos(\theta+\phi) &= 0, \end{aligned}$$

where (x, y) are the Cartesian coordinates of the rear wheel, given by

$$\begin{aligned} x &= x_f - \ell\cos\theta \\ y &= y_f - \ell\sin\theta. \end{aligned}$$

Computing $\dot{x}$, $\dot{y}$ from these formulas and plugging them in the rear wheel constraint leads to

$$\dot{x}_f\sin\theta - \dot{y}_f\cos\theta + \ell\,\dot{\theta} = 0.$$

Being $\boldsymbol{q} = (x_f, y_f, \theta, \phi)$, the Pfaffian constraints are written in matrix form as

$$\begin{pmatrix} \sin\theta & -\cos\theta & \ell & 0 \\ \sin(\theta+\phi) & -\cos(\theta+\phi) & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x}_f \\ \dot{y}_f \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \boldsymbol{A}^T(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{0}.$$

A basis $\{\boldsymbol{g}_1, \boldsymbol{g}_2\}$ for the 2-dimensional null space of $\boldsymbol{A}^T$ can be immediately written as

$$\boldsymbol{g}_1(\boldsymbol{q}) = \begin{pmatrix} \cos(\theta+\phi) \\ \sin(\theta+\phi) \\ (\sin\phi)/\ell \\ 0 \end{pmatrix} \qquad \boldsymbol{g}_2(\boldsymbol{q}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The kinematic model is then

$$\begin{aligned} \dot{x}_f &= \cos(\theta+\phi)\,v \\ \dot{y}_f &= \sin(\theta+\phi)\,v \\ \dot{\theta} &= \frac{\sin\phi}{\ell}\,v \\ \dot{\phi} &= \omega, \end{aligned}$$

where v and ω are clearly the driving and steering velocity of the bicycle, both generated at the front wheel.

To study the controllability of the above kinematic model, compute

$$[\boldsymbol{g}_1, \boldsymbol{g}_2](\boldsymbol{q}) = \begin{pmatrix} \sin(\theta+\phi) \\ -\cos(\theta+\phi) \\ -(\cos\phi)/\ell \\ 0 \end{pmatrix} = \boldsymbol{g}_3(\boldsymbol{q}) \qquad \text{and} \qquad [\boldsymbol{g}_1, \boldsymbol{g}_3](\boldsymbol{q}) = \begin{pmatrix} -(\sin\theta)/\ell \\ (\cos\theta)/\ell \\ 0 \\ 0 \end{pmatrix} = \boldsymbol{g}_4(\boldsymbol{q})$$

We have

$$\text{rank}\,(\boldsymbol{g}_1\ \boldsymbol{g}_2\ \boldsymbol{g}_3\ \boldsymbol{g}_4) = \text{rank}\,(\boldsymbol{g}_1\ \boldsymbol{g}_3\ \boldsymbol{g}_4\ \boldsymbol{g}_2) = \text{rank} \begin{pmatrix} \cos(\theta+\phi) & \sin(\theta+\phi) & -(\sin\theta)/\ell & 0 \\ \sin(\theta+\phi) & -\cos(\theta+\phi) & (\cos\theta)/\ell & 0 \\ (\sin\phi)/\ell & -(\cos\phi)/\ell & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 4$$

because the determinant of the upper 3×3 block is $1/\ell^2$. The kinematic model is then controllable.

(b) From the first two equations of the kinematic model, we obtain

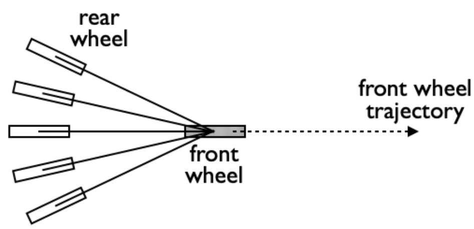
$$\theta = \arctan\frac{\dot{y}_f}{\dot{x}_f} - \phi \qquad \Rightarrow \qquad \dot{\theta} = \frac{\ddot{y}_f\dot{x}_f - \dot{y}_f\ddot{x}_f}{\dot{x}_f^2 + \dot{y}_f^2} - \dot{\phi}.$$

From the third equation of the kinematic model we get

$$\phi = \arcsin \frac{\ell\, \dot\theta}{v} \qquad \Rightarrow \qquad \phi = \arcsin \frac{\ell}{v} \left(\frac{\ddot y_f \dot x_f - \dot y_f \ddot x_f}{\dot x_f^2 + \dot y_f^2} - \dot\phi \right).$$

having used the above expression of $\dot\theta$. Since this is a differential equation, we conclude that ϕ (and consequently also θ) cannot be reconstructed algebraically from x_f, y_f and their derivatives. Therefore, x_f, y_f are *not* flat ouputs.

This result can be interpreted as follows. For the same trajectory of the front wheel, there are infinite possible trajectories of the rear wheel, depending on its initial condition (see the following figure). Indeed, this is consistent with the fact that ϕ is governed by a differential equation.



(c) The first-order time derivatives of the outputs x_f, y_f are given by the first two equations of the kinematic model. Since only v appears in them, we compute the second-order derivatives:

$$\begin{aligned} \ddot x_f &= -\sin(\theta+\phi)(\dot\theta+\dot\phi)\, v + \cos(\theta+\phi)\, \dot v \\ \ddot y_f &= \cos(\theta+\phi)(\dot\theta+\dot\phi)\, v + \sin(\theta+\phi)\, \dot v. \end{aligned}$$

Setting $\dot v = a$ (dynamic extension), using the expressions of $\dot\theta$ and $\dot\phi$ given by the kinematic model, and rearranging terms one obtains

$$\begin{pmatrix} \ddot x_f \\ \ddot y_f \end{pmatrix} = \begin{pmatrix} -\sin(\theta+\phi)\dfrac{\sin\phi}{\ell}\, v^2 \\ \cos(\theta+\phi)\dfrac{\sin\phi}{\ell}\, v^2 \end{pmatrix} + \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi)\, v \\ \sin(\theta+\phi) & \cos(\theta+\phi)\, v \end{pmatrix} \begin{pmatrix} a \\ \omega \end{pmatrix} = \boldsymbol{b}(\boldsymbol{q}, v) + \boldsymbol{T}(\boldsymbol{q}, v) \begin{pmatrix} a \\ \omega \end{pmatrix}.$$

The decoupling matrix $\boldsymbol{T}$ is obviously invertible when $v \neq 0$, so under this assumption we can let

$$\begin{pmatrix} a \\ \omega \end{pmatrix} = \boldsymbol{T}^{-1}(\boldsymbol{q}, v) \left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \boldsymbol{b}(\boldsymbol{q}, v) \right), \tag{1}$$

thus obtaining a second-order linear mapping between the output and the new inputs u_1, u_2:

$$\begin{aligned} \ddot x_f &= u_1 \\ \ddot y_f &= u_2. \end{aligned}$$

Globally exponential tracking of the desired trajectory $x_f^*(t)$, $y_f^*(t)$ is then guaranteed by the following PD+feedforward control law

$$\begin{aligned} u_1 &= \ddot x_f^* + k_{p1}(x_f^* - x_f) + k_{d1}(\dot x_f^* - \dot x_f) \\ u_2 &= \ddot y_f^* + k_{p2}(y_f^* - y_f) + k_{d2}(\dot y_f^* - \dot y_f), \end{aligned}$$

as long as the control gains k_{p1}, k_{d1}, k_{p2}, k_{d2} are positive. Note the following points.

- To compute u_1, u_2 we need real-time measurements of the outputs x_f, y_f (and their first-order derivatives). These quantities can be directly available or, more often, reconstructed from measurements of the original state variables x, y, θ (and numerical differentiation).
- The expression of the control inputs a, ω is found by plugging the above expressions of u_1, u_2 into (1); measurements of the θ, ϕ as well as of the additional state variable v are needed for this computation.
- Finally, the driving velocity input v is obtained by integrating the acceleration input a. This means that the proposed controller is inherently dynamic.

Solution of Problem 2

(a) The (2,4) chained form is

$$\begin{aligned}\dot z_1 &= v_1\\ \dot z_2 &= v_2\\ \dot z_3 &= z_2\, v_1\\ \dot z_4 &= z_3\, v_1.\end{aligned}$$

From the last and the first model equation, we get

$$z_3 = \frac{\dot z_4}{v_1} = \frac{\dot z_4}{\dot z_1} \qquad \Rightarrow \qquad \dot z_3 = \frac{\ddot z_4 \dot z_1 - \dot z_4 \ddot z_1}{\dot z_1^2}.$$

The equation on the left is the reconstruction formula for z_3. As for z_2, the third and the first model equation lead to the reconstruction formula

$$z_2 = \frac{\dot z_3}{v_1} = \frac{\dot z_3}{\dot z_1} = \frac{\ddot z_4 \dot z_1 - \dot z_4 \ddot z_1}{\dot z_1^3},$$

where we have used the previous expression for $\dot z_3$. The reconstruction formulas for the inputs are

$$\begin{aligned}v_1 &= \dot z_1\\ v_2 &= \dot z_2 = \frac{d}{dt}\frac{\ddot z_4 \dot z_1 - \dot z_4 \ddot z_1}{\dot z_1^3} = \ldots\end{aligned}$$

All the reconstruction formulas are algebraic functions of z_1, z_4 and their time derivatives; we can therefore conclude that z_1, z_4 are indeed flat outputs.

(b) Path planning from $\boldsymbol{z}_i = (z_{1i}, z_{2i}, z_{3i}, z_{4i})$ to $\boldsymbol{z}_f = (z_{1f}, z_{2f}, z_{3f}, z_{4f})$ may be performed by interpolating the flat outputs with the appropriate boundary conditions. Assume that the range for the path parameter s is $[0,1]$. The boundary conditions for z_1 are

$$z_1(0) = z_{1i} \quad \text{and} \quad z_1(1) = z_{1f}.$$

A linear polynomial will then suffice:

$$z_1(s) = z_{1i} + s(z_{1f} - z_{1i}) \quad s \in [0,1].$$

Accordingly, it is

$$z_1'(s) = z_{1f} - z_{1i} \quad \text{and} \quad z_1''(s) = 0, \quad \forall s.$$

As for z_4, in addition to the boundary conditions

$$z_4(0) = z_{4i} \quad \text{and} \quad z_4(1) = z_{4f}$$

we must consider those on its derivatives coming from the initial and final values of z_2, z_3. In particular, isolating z_4' from the reconstruction formula[1] for z_3, we get $z_4' = z_3 z_1' = z_3(z_{1f} - z_{1i})$, from which

$$z_4'(0) = z_{3i}(z_{1f} - z_{1i}) \quad \text{and} \quad z_4'(1) = z_{3f}(z_{1f} - z_{1i}).$$

Finally, isolating z_4'' from the reconstruction formula for z_2 gives $z_4'' = (z_2(z_1')^3 + z_4' z_1'')/z_1' = z_2(z_1')^2$, from which we get

$$z_4''(0) = z_{2i}(z_{1f} - z_{1i})^2 \quad \text{and} \quad z_4''(1) = z_{2f}(z_{1f} - z_{1i})^2.$$

There are a total of six boundary conditions for z_4 and its derivatives, suggesting the use of a 5-th order polynomial:

$$z_4(s) = as^5 + bs^4 + cs^3 + ds^2 + es + f, \quad s \in [0,1].$$

The six unknown coefficients $a, \ldots, f$ can be found by solving the linear system built with the boundary conditions.

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[1] Since we are planning a geometric path, the geometric model applies in place of the kinematic model. Accordingly, time derivatives in the reconstruction formulas are replaced by derivatives w.r.t. s, denoted by $'$, $''$ and so on.

Solution of Problem 3

With the state $\boldsymbol{x} = (x, z, \theta, \dot{x}, \dot{z}, \dot{\theta}) = (x_1, \ldots, x_6)$, the continuous-time model of the system is written as

$$\begin{aligned}
\dot{x}_1 &= x_4 \\
\dot{x}_2 &= x_5 \\
\dot{x}_3 &= x_6 \\
\dot{x}_4 &= f_1 \cos x_3 - f_2 \sin x_3 \\
\dot{x}_5 &= -g + f_1 \sin x_3 + f_2 \cos x_3 \\
\dot{x}_6 &= -d\, f_2
\end{aligned}$$

Using Euler integration, a discrete-time version of this model is written as

$$\begin{aligned}
x_{1,k+1} &= x_{1,k} + x_{4,k}\, T \\
x_{2,k+1} &= x_{2,k} + x_{5,k}\, T \\
x_{3,k+1} &= x_{3,k} + x_{6,k}\, T \\
x_{4,k+1} &= x_{4,k} + f_{1,k} \cos x_{3,k}\, T - f_{2,k} \sin x_{3,k}\, T \\
x_{5,k+1} &= x_{5,k} - g\, T + f_{1,k} \sin x_{3,k}\, T + f_{2,k} \cos x_{3,k}\, T \\
x_{6,k+1} &= x_{6,k} - d\, f_{2,k}\, T.
\end{aligned}$$

This motion model is assumed to be perturbed by a white gaussian noise with zero mean and known covariance.

As for the measurement model, we have a total of four measurements coming from the sensors at each sampling instant. The first is the bearing angle of the beacon

$$y_{1k} = \text{atan2}(z_b - z_k, x_b - x_k) - \theta_k.$$

The second and third measurement are the CoM velocities $\dot{x}$ and $\dot{y}$:

$$y_{2k} = x_{4,k} \qquad y_{3k} = x_{5,k},$$

while the fourth measurement is the robot angular velocity $\dot{\theta}$:

$$y_{4k} = x_{6,k}.$$

The measurement model is therefore

$$\boldsymbol{y} = \begin{pmatrix} y_{1k} \\ y_{2k} \\ y_{3k} \\ y_{4k} \end{pmatrix}$$

with the previous formulas providing the expression of each component as a function of the state variables. This model is also assumed to be perturbed by a white gaussian noise with zero mean and known covariance.

The rest of the solution is straightforward: linearize the motion and measurement models (only the fourth and fifth equations of the former and the first equation of the latter are actually nonlinear) and then write the EKF equations. Note that the (nominal) force inputs $f_{1,k}$ and $f_{2,k}$ coming from the control module will be needed in the prediction stage.