Stabilization of Nonlinear Systems via State Feedback

G. Oriolo

Sapienza University of Rome

Introduction

we consider a generic time-invariant **nonlinear** dynamical system

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= g(x) \end{aligned}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^p$, and output $y \in \mathbb{R}^q$

stabilization via state feedback

design a control law $u = k(x)$ such that the closed-loop system

$$\dot{x} = f(x, k(x))$$

has a given state x_d as an **asymptotically stable** equilibrium point

- x_d is specified by the control problem and represents a **desired operating state** for the system: for example, an attitude for a satellite, a pose in space for a robotic manipulator, a temperature for a climate-control system
- x_d need not be an equilibrium point of the open-loop system; however, it **must** become one for the closed-loop system
- in the following, we assume that x_d is the **origin**; indeed, it is always possible to reduce to this case by applying the coordinate translation $z = x - x_d$

- for a **linear** system $\dot{x} = Ax + Bu$, a state feedback is $u = K\,x$; the closed-loop system becomes

$$\dot{x} = Ax + BK\,x = (A + BK)x$$

as is well known, the problem of stabilization via state feedback is solvable if the pair (A, B) is **stabilizable**, i.e., if it is completely controllable or if any uncontrollable eigenvalues have negative real part

- a feedback of the form $u = k(x)$ is called **static** because it represents a memoryless controller; we speak of **dynamic** feedback when the control is itself the output of a dynamical system driven by the state x:

$$\begin{aligned} \dot{\xi} &= \phi(\xi, x) \\ u &= k(\xi) \end{aligned}$$

- state feedback assumes that all components of x can be measured; when this is not possible, one resorts to **output feedback**, which may be static ($u = k(y)$) or, more often, dynamic:

$$\begin{aligned} \dot{\xi} &= \phi(\xi, y) \\ u &= k(\xi) \end{aligned}$$

for example, this is the case in which the controller includes an asymptotic observer used to reconstruct the state

Stabilization via linear approximation

Lyapunov **basic idea**

compute the **linear approximation** of the system about the origin and stabilize it via **linear** feedback; by Lyapunov indirect method, the origin will be **locally** asymptotically stable for the nonlinear system

ex: consider the scalar system

$$\dot{x} = a\,x^2 + u$$

containing the parameter $a > 0$; its linear approximation about the origin is $\dot{x} = u$, which is obviously stabilized by the linear feedback $u = -k\,x$ with $k > 0$

applying this control to the nonlinear system, the closed loop becomes

$$\dot{x} = a\,x^2 - k\,x \qquad (*)$$

which, by Lyapunov indirect method, has the origin as an asymptotically stable equilibrium

- the asymptotic stability property is **local**: indeed, system $(*)$ has another equilibrium at $x = k/a$, and diverges for $x > k/a \Rightarrow$ the region of attraction is $\Omega = \{x : x < k/a\}$
- to achieve convergence from any set $S = \{x : |x| < r\}$, it is enough to set $k > a\,r$; the stability is **semi-global**, in the sense that by modifying the controller parameters (here k) one can include in Ω any neighborhood of the origin
- the stability obtained is not, however, global, since once k is chosen there exist states (here $\{x : x > k/a\}$) from which convergence does not occur ■

let's apply the same approach to a generic time-invariant nonlinear system

$$\dot{x} = f(x, u)$$

under the hypothesis that $(x = 0, u = 0)$ is an equilibrium, i.e., that the origin is an unforced equilibrium point

the linear approximation of the system about $(x = 0, u = 0)$ is

$$\dot{x} = \left.\frac{\partial f(x,u)}{\partial x}\right|_{x=0,u=0} (x - 0) + \left.\frac{\partial f(x,u)}{\partial u}\right|_{x=0,u=0} (u - 0) = Ax + Bu$$

if the pair (A, B) is **stabilizable**, one can design a linear state feedback $u = K\,x$ such that the eigenvalues of $(A + BK)$ have negative real part, and the linear approximation is therefore (globally and exponentially) asymptotically stable

$\Rightarrow$ $u = K\,x$ makes the origin (locally) **asymptotically stable** for the nonlinear system

- if the pair (A, B) is **not stabilizable**, there is no linear feedback that stabilizes the linear approximation; **however, one cannot exclude** that there exists a feedback able to stabilize the nonlinear system, and such feedback may even be linear

  ex: $\dot{x} = u^3$, whose linear approximation is $\dot{x} = 0$, is stabilized by $u = -x$

- if (A, B) is stabilizable, this approach also provides an **estimate of the domain of attraction**, since it is easy to write a Lyapunov function for the nonlinear system starting from the linear approximation; to this end, the following result is useful

Theorem

a linear system $\dot{x} = Ax$ is asymptotically stable if and only if, for any given symmetric and positive definite matrix Q, the following **Lyapunov equation** $$PA + A^T P = -Q$$ admits a unique symmetric and positive definite solution in the unknown P

proof (sufficiency) it is an application of Lyapunov direct stability method; indeed, taking as Lyapunov candidate

$$V(x) = \frac{1}{2} x^T P x$$

which is PD by hypothesis, we have

$$\dot{V} = x^T P \dot{x} = x^T P A x = \frac{1}{2}(x^T P A x + x^T P A x) = \frac{1}{2}\left(x^T (PA + A^T P) x\right) = -\frac{1}{2} x^T Q x$$

which is ND by hypothesis (we used $x^T P A x = (x^T P A x)^T = x^T A^T P x$) ∎

in our case, since the closed-loop linear approximation $\dot{x} = (A + BK)x$ is asymptotically stable, it admits as Lyapunov function

$$V(x) = \frac{1}{2} x^T P x$$

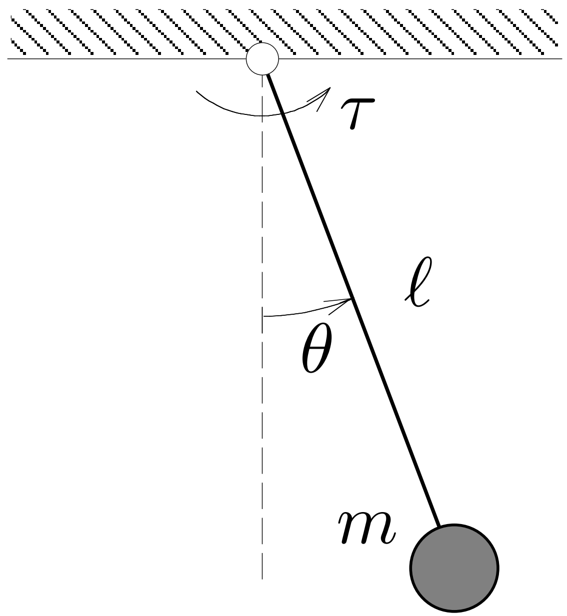
where P is the unique symmetric and PD solution of the corresponding Lyapunov equation

$$P(A + BK) + (A + BK)^T P = -Q$$

with Q arbitrary but symmetric and PD (for example, $Q = I$)

. . . and V is a Lyapunov function **also for the nonlinear system!**

ex: pendulum with joint torque actuator



$$m\,\ell^2\,\ddot{\theta} + d\,\dot{\theta} + m\,g\,\ell\,\sin\theta = \tau$$

letting $x = (x_1, x_2) = (\theta, \dot{\theta})$ and $\tau = u$ the state-space equation is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\sin x_1 - b\,x_2 + c\,u \end{aligned}$$

where $a = g/\ell$, $b = d/m\,\ell^2$, $c = 1/m\,\ell^2$ $(a, b, c > 0)$

suppose we wish to stabilize the pendulum at a **generic** angle θ_d; the desired equilibrium point is therefore $x_d = (x_{1d}, x_{2d}) = (\theta_d, 0)$

we perform the coordinate transformation $z = x - x_d = (x_1 - \theta_d, x_2)$

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= -a\sin(z_1 + \theta_d) - b\, z_2 + c\, u\end{aligned}$$

to make the origin $z_1 = 0, z_2 = 0$ an unforced equilibrium point, set $u = u_{\text{fb}} + u_{\text{ff}}$, where u_{fb} is the **feedback component** and u_{ff} is the **feedforward component**

$u_{\text{fb}} = Kz$ vanishes automatically at the origin, and therefore u_{ff} has the task of making that point an equilibrium:

$$-a\sin\theta_d + c\, u_{\text{ff}} = 0 \qquad \text{hence} \qquad u_{\text{ff}} = \frac{a}{c}\sin\theta_d = m\, g\ell\sin\theta_d$$

that is, u_{ff} is the torque required to balance gravity when the pendulum is at θ_d

the closed-loop system is therefore

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= -a\,(\sin(z_1 + \theta_d) - \sin\theta_d) - b\, z_2 + c\, u_{\text{fb}}\end{aligned}$$

which finally has $z = 0$, $u_{\text{fb}} = 0$ as an equilibrium point

the linear approximation of the system is therefore characterized by the matrices

$$\begin{aligned}A &= \left.\frac{\partial f(z, u_{\text{fb}})}{\partial z}\right|_{z=0, u_{\text{fb}}=0} = \left.\begin{pmatrix} 0 & 1 \\ -a\cos(z_1 + \theta_d) & -b \end{pmatrix}\right|_{z=0, u_{\text{fb}}=0} = \begin{pmatrix} 0 & 1 \\ -a\cos\theta_d & -b \end{pmatrix} \\ B &= \left.\frac{\partial f(z, u_{\text{fb}})}{\partial u_{\text{fb}}}\right|_{z=0, u_{\text{fb}}=0} = \begin{pmatrix} 0 \\ c \end{pmatrix}\end{aligned}$$

the controllability matrix is

$$\begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 0 & c \\ c & -b\,c \end{pmatrix}$$

it is therefore possible to arbitrarily assign the closed-loop eigenvalues of the linear approximation; it is easy to verify that with the linear feedback

$$u_{\text{fb}} = Kz = \begin{pmatrix} k_1 & k_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = k_1 z_1 + k_2 z_2$$

the eigenvalues of $A + BK$ have negative real part provided that $k_1 < \frac{a}{c} \cos \theta_d$ and $k_2 < \frac{b}{c}$

$\Rightarrow$ under these assumptions, the torque

$$u = u_{\text{fb}} + u_{\text{ff}} = k_1\, z_1 + k_2\, z_2 + \frac{a}{c} \sin \theta_d = k_1(\theta - \theta_d) + k_2\, \dot{\theta} + m\, g\ell \sin \theta_d$$

renders the point $(\theta_d, 0)$ (locally) asymptotically stable for the pendulum

- note the **physical interpretation** of the term u_{fb}, which simulates an angular spring that pulls the pendulum back to the position θ_d and a viscous damper that dissipates energy

- the domain of attraction will depend in a **crucial** way on the choice of k_1 and k_2; it is possible to estimate its size by using, as a Lyapunov candidate for the nonlinear system, a Lyapunov function for the linear approximation setting, for example, $a = c = 1$, $b = 0$, $\theta_d = \pi/2$ and $k_1 = k_2 = -1$ we obtain

$$A + BK = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

and the corresponding Lyapunov equation (for $Q = I$)

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

admits the symmetric and positive definite solution

$$P = \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 1 \end{pmatrix}$$

hence we can use the following as Lyapunov function for the nonlinear system

$$V(x) = \frac{1}{2} x^T \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 1 \end{pmatrix} x$$

at this point, identify the set where $\dot{V}(x)$ is ND, and take any level curve contained in that set; the region inside this level curve is an estimate of the domain of attraction for the (linear) controller considered

Stabilization via exact linearization: Basics

the main limitation of the stabilization technique via linear approximation is that convergence is guaranteed only within a domain of attraction, which may be more or less large; this may be unacceptable in practice

ex: for the scalar system

$$\dot{x} = a\,x^2 + u$$

we have seen that the linear feedback $u = -k\,x$ with $k > 0$ makes the origin asymptotically stable, with region of attraction $\Omega = \{x : x < k/a\}$

consider instead the following **nonlinear** control law

$$u = -a\,x^2 - k\,x$$

which **cancels** the nonlinear term $a\,x^2$ and leads to the following closed-loop system

$$\dot{x} = -k\,x$$

the system is **exactly** linear, and the origin is therefore a **globally** asymptotically stable equilibrium

this control law has two components: one ($-a\,x^2$) is in charge of **exactly linearizing** the closed-loop dynamics, and the other ($-k\,x$) of **stabilizing** that dynamics ■

ex: consider again the pendulum with joint actuator

$$\begin{aligned}\dot{z}_1 &= z_2\\ \dot{z}_2 &= -a\,\sin(z_1+\theta_d) - b\,z_2 + c\,u\end{aligned}$$

for which we have already performed the coordinate transformation $z = x - x_d = (x_1 - \theta_d, x_2)$ needed to shift the desired equilibrium point to the origin

inspection of the second differential equation, which is the only one containing nonlinear terms, suggests the following choice for u

$$u = \frac{a}{c}\,\sin(z_1+\theta_d) + \frac{v}{c}$$

the closed-loop dynamics becomes linear and completely controllable

$$\begin{aligned}\dot{z}_1 &= z_2\\ \dot{z}_2 &= -b\,z_2 + v\end{aligned}$$

it is therefore possible to stabilize it **globally** at the origin via the new input v

$$v = k_1\,z_1 + k_2\,z_2$$

with k_1 and k_2 chosen so as to assign arbitrary eigenvalues; we thus have

$$u = \frac{a}{c}\sin\theta + \frac{1}{c}\left(k_1(\theta-\theta_d) + k_2\dot{\theta}\right)$$

in which **all** terms are in feedback (in particular, at the equilibrium the first term automatically becomes the torque needed to balance gravity) ■

then, it is natural to ask

> how general is the idea of canceling nonlinearities via feedback? is there a **structural property** of systems that guarantees this possibility?

we are **certainly** able to do this if the state equation has the following structure

$$\dot{x} = f(x, u) = A\,x + B\,\beta(x)\,(u - \alpha(x))$$

with $\beta(x)$ a nonsingular matrix on a domain containing the origin (note that the two previous examples have exactly this structure)

in fact, it is sufficient to set

$$u = \alpha(x) + \beta^{-1}(x)v$$

to obtain the linear system

$$\dot{x} = A\,x + B\,v$$

which can be stabilized by setting $v = K\,x$ (if the pair (A, B) is stabilizable); the overall feedback law is

$$u = \alpha(x) + \beta^{-1}(x)K\,x$$

note that it is **nonlinear**!

if the system model does **not** have the above structure, it may be possible to put it into that form via a **coordinate transformation**

ex: for the system

$$\begin{aligned}\dot{x}_1 &= a \sin x_2 \\ \dot{x}_2 &= -x_1^2 + u\end{aligned}$$

it is clear that it is not possible to cancel the nonlinearity $a \sin x_2$ through u

consider however the following coordinate transformation

$$\begin{aligned}z_1 &= x_1 \\ z_2 &= a \sin x_2 = \dot{x}_1\end{aligned}$$

we have

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= a \cos x_2 (-x_1^2 + u)\end{aligned}$$

it is now possible to cancel the nonlinearity with the feedback law

$$u = x_1^2 + \frac{1}{a \cos x_2} v$$

which is well defined for $-\pi/2 < x_2 < \pi/2$

note that the coordinate transformation $z = T(x)$ is well posed, since it can be inverted as follows

$$\begin{aligned} x_1 &= z_1 \\ x_2 &= \arcsin\left(\frac{z_2}{a}\right) \end{aligned}$$

in the set $-a < z_2 < a$

moreover, both the transformation $T(\cdot)$ and its inverse $T^{-1}(\cdot)$ are continuously differentiable
$\Rightarrow$ we say that $T(\cdot)$ is a **diffeomorphism** ■

the properties of this example can be extrapolated into the following definition

a nonlinear system

$$\dot{x} = f(x, u)$$

is said to be **input-state linearizable** if there exists a diffeomorphism $z = T(x)$, defined on a domain D_x containing the origin, that puts the system in the form

$$\dot{z} = Az + B\beta(x)\,(u - \alpha(x))$$

with the matrix $\beta(x)$ nonsingular in D_x

input-state linearizable systems can therefore be effectively controlled (for example, globally exponentially stabilized at a point) through a **coordinate transformation** and a **static state feedback** which has a dual role: (1) cancel the system nonlinearities (2) control the linearized system

- there is also the possibility of achieving input-state linearization of a system via coordinate transformation and **dynamic** state feedback; the class of systems that can be linearized in this way is **larger** than that of systems linearizable with static feedback

- in the case where the control problem is formulated at the level of the system **outputs** (for example, in reference-output tracking problems), one may try to achieve **input-output linearization**, again using a coordinate transformation plus static or dynamic state feedback

- a drawback of exact linearization is that canceling nonlinearities requires **exact** knowledge of the model parameters; for example, in the case of the system $\dot{x} = ax^2 + u$ the control law computed via exact linearization (slide 11) was

$$u = -a\,x^2 - k\,x$$

which contains the parameter a; instead, the control law computed via linear approximation (slide 4) was

$$u = -k\,x$$

$\Rightarrow$ for controllers designed through the exact linearization method there is a potential problem of **robustness** with respect to parameter variations, which must be analyzed carefully

- another drawback of the exact linearization approach is that it may lead to the cancellation of **nonlinear** terms that are actually **beneficial** for stabilization

  ex: consider the nonlinear scalar system

$$\dot{x} = a\,x - b\,x^3 + u \qquad a, b > 0$$

  a controller based on the exact linearization philosophy is the following

$$u = -k\,x + b\,x^3 \qquad k > a$$

  however, the nonlinear term $-b\,x^3$ can be interpreted as a **nonlinear elastic force** that pushes the state toward the origin; indeed, the simple linear controller

$$u = -k\,x \qquad k > a$$

  leads to the closed-loop system

$$\dot{x} = -(k-a)x - b\,x^3$$

  the origin is GAS, and trajectories converge faster than those of $\dot{x} = -(k-a)x$ ■

  a possible consequence of this unnecessary cancellation, due to the mathematical (rather than physical) nature of the exact linearization approach, is a **higher control effort** (in the example, $u = -k\,x + b\,x^3$ takes on (absolute) values much larger than $u = -k\,x$ when far from the origin)

  ⇒ it is sometimes preferable to avoid linearization and design the controller using the **direct Lyapunov method**, which lends itself better to a physical interpretation