Passivity, Feedback Equivalence, and the Global Stabilization of Minimum Phase Nonlinear Systems

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Abstract—**In this paper, we derive conditions under which a nonlinear system can be rendered passive via smooth state feedback and we show that, as in the case linear systems, this is possible if and only if the system in question has relative degree 1 and is weakly minimum phase. Then, we prove that weakly minimum phase nonlinear systems having relative degree 1 can be globally asymptotically stabilized by smooth state feedback, provided that suitable controllability-like rank conditions are satisfied. This result incorporates and extends a number of stabilization schemes recently proposed in the literature for global asymptotic stabilization of certain classes of nonlinear systems.**

I. Introduction

THE feedback stabilization of nonlinear systems has occupied a central role in the nonlinear systems literature for at least three decades. Widely recognized as an important problem in its own right, feedback stabilization is also important as a preliminary step in achieving additional control objectives, e.g., asymptotic tracking, disturbance attenuation, etc. This role hints at a broader version of this important problem, a version which would require facilitating further analysis of the overall behavior of more general interconnections involving the system to be controlled in one of its inner loops. Analysis of this broader problem was one of the starting points in the 1960's for the development of the operator theoretic approach to the analysis of input–output systems.

For example, the "small gain" theorem yields a number of interesting corollaries concerning the overall stability of various important interconnections of linear and nonlinear systems. Also, the notion of "passivity" of an input–output system, motivated by the dissipation of energy across resistors in an electrical circuit, has been widely used in order to analyze stability of a general class of interconnected nonlinear systems (see, e.g., [6], [20]–[23], [25]). Beginning in the early 1970's, passivity was also studied for state-space representations of nonlinear systems, allowing for a more geometric interpretation of notions such as available, stored, and dissipated energy in terms of Lyapunov functions [26]–[27]. This point of view, leading to a Lyapunov-theoretic counterpart of many stability results developed within an input–output point of view approach as well as to a nonlinear form of the Kalman–Yacubovitch–Popov lemma, has since been specifically developed in the series of papers [9]–[13]. In particular, in addition to basic stability results, we now know a fairly complete answer to the fundamental question: *when is a finite-dimensional nonlinear system passive?*

Passive systems, like linear circuits containing only positive resistors, are stable. However, there are many engineering systems (e.g., high-performance aircraft or even some more exotic circuits) which are even designed to be unstable for some range of initial conditions in order to take advantage of improved performance for other ranges of initial data. Such systems provide examples of the desirability of using feedback to stabilize unstable systems and in fact, during the last two decades, there has been a considerable attention, in the literature on geometric control of nonlinear systems, to the problem of stabilization via state feedback. Among the major trends which focus on this problem, there are three somewhat distinct approaches which are related to the problems and techniques presented in this paper: global asymptotic stabilization of Lyapunov stable systems under suitable "controllability" conditions, feedback linearization of nonlinear systems, and asymptotic stabilization of the nonlinear equivalent of minimum phase linear systems.

As one might hope, a synthesis of concepts and techniques drawn from the theory of passive systems and from the geometric nonlinear control theory leads to a more powerful methodology for the design of nonlinear feedback systems. In particular, in view of the role played by the concept of passivity not just in terms of system stability but also in the analysis of the stability of interconnected feedback systems, another fundamental question arises, which appears to have a variety of rather interesting consequences: *when can a finite-dimensional nonlinear system be rendered passive via state feedback?*

In this paper, we shall address this question and provide a rather complete answer in terms of geometric nonlinear system theory. Perhaps surprisingly, the characterization we obtain is a nonlinear enhancement of some very classical

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facts concerning passivity in a linear system. For example, passive nonlinear systems enjoy the nonlinear analog of the minimum phase property (expressed in terms of the system's zero dynamics). And, in terms of such a nonlinear feedback invariant we are able to solve, under mild regularity assumptions, the problem of identifying those nonlinear systems which are *feedback equivalent* to *passive* systems. In one of its forms, this problem is a generalization of the problem of feedback equivalence to *linear* systems, in a precise technical sense. But more importantly, because passive systems represent a class of systems for which feedback analysis and design is comparatively more simple, more intuitive, and better understood, the problem of feedback equivalence to a passive nonlinear system is a less stringent yet very appealing version of the problem of feedback equivalence to a linear system.

The main results of this paper are contained in Section IV, in which we solve the problem of feedback equivalence to a passive system under some mild regularity hypotheses, which can be relaxed in certain circumstances. In Section II, we fix our notation and nomenclature, reviewing some of the basic definitions and concepts from the theory of dissipative systems as developed in [26]–[27] and [10]–[13]. In Section III, we revisit some basic results concerning the asymptotic stabilization of passive systems via static output feedback (see, e.g., [10]). In particular, we show that the frequently assumed property of observability can be weakened by requiring only detectability, and present a criterion implying detectability, which is stated in terms usually associated with accessibility and controllability criteria. This somehow intriguing relationship between accessibility conditions and a dual property of detectability reposes on the nonlinear version of Kalman–Yacubovitch–Popov lemma (see [10]), which of course relates the system output to the control vector field. As a consequence, we also obtain as corollaries a series of previous results on global feedback stabilization of Lyapunov stable nonlinear systems which satisfy certain controllability conditions [7], [15], [16], [18].

In Section V, we combine the stability results for passive systems discussed in Section III with the feedback equivalence criteria derived in Section IV to obtain a further class of state feedback stabilization methods for various interconnections of nonlinear systems. In particular, we give a new proof of previous global stabilization results for minimum phase nonlinear systems and a rather powerful extension for feedback stabilization of "weakly" minimum phase nonlinear systems. These results also yield a fully nonlinear version of a stabilization criterion for a cascade-interconnected configuration which has recently attracted some attention in the geometric nonlinear control literature [17], [24].

II. Dissipative and Passive Systems.

In this paper, we consider nonlinear systems described by equations of the form

$$\dot{x} = f(x) + g(x)u \tag{2.1a}$$

$$y = h(x) \tag{2.1b}$$

with state space $X = \mathbf{R}^n$, set of input values $U = \mathbf{R}^m$ and set of output values $Y = \mathbf{R}^m$. The set $\mathscr{U}$ of admissible inputs consists of all U-valued piecewise continuous functions defined on $\mathbf{R}$. f and the m columns of g are smooth (i.e., C^∞) vector fields and h is a smooth mapping. We suppose that the vector field f has at least one equilibrium; thus, without loss of generality, after possibly a coordinates shift, we can assume $f(0) = 0$ and $h(0) = 0$.

We review in this section a number of basic concepts related to the notions of dissipativity and passivity; the reader is referred to [26] and [12] for additional details. Let w be a real-valued function defined on $U \times Y$, called the *supply rate*. We assume that for any $u \in \mathscr{U}$ and for any $x^\circ \in X$, the output $y(t) = h(\Phi(t, x^\circ, u))$ of (2.1) is such that $w(s) = (w(u(s), y(s))$ satisfies

$$\int_0^t |w(s)|\, ds < \infty \qquad \text{for all } t \geq 0.$$

Definition 2.1: A system Σ of the form (2.1) with supply rate w is said to be *dissipative* if there exists a C° nonnegative function $V\colon X \to \mathbf{R}$, called the *storage function*, such that for all $u \in \mathscr{U}$, $x^\circ \in X$, $t \geq 0$

$$V(x) - V(x^\circ) \leq \int_0^t w(s)\, ds$$

where $x = \Phi(t, x^\circ, u)$. △

The above inequality is called the *dissipation inequality*. The next definition characterizes the notion of *available storage*, which plays an important role in determining whether or not a given system is dissipative.

Definition 2.2: The *available storage*, denote V_a, of a system Σ with supply rate w is the function V_a: $X \to \mathbf{R}$ defined by

$$V_a(x) = \sup_{\substack{x^\circ = x \\ u \in \mathscr{U} \\ t \geq 0}} \left\{ -\int_0^t w(s)\, ds \right\}. \qquad \triangle$$

Note that the available storage, whenever defined, is nonnegative, since $V_a(x)$ is the supremum over a set of numbers which contains the zero element. The following statement illustrates the properties of such a function.

Proposition 2.3 [26]: If a system Σ with supply rate w is dissipative, the available storage $V_a(x)$ is finite for each $x \in X$. Moreover, any possible storage function V satisfies

$$0 \leq V_a(x) \leq V(x)$$

for each $x \in X$, and if V_a is C°, then V_a itself is a possible storage function. Conversely, if $V_a(x)$ is finite for each $x \in X$ and C°, then the system Σ is dissipative.

Throughout the paper, we shall be interested in studying dissipative systems with supply rate given by the inner product.

$$w = \langle u, y \rangle = y^{\mathrm{T}} u$$

(where the superscript "T" denotes transpose). For convenience, we characterize this choice by means of a separate definition.

Definition 2.4: A system Σ is said to be *passive* if it is dissipative with supply rate $w = \langle u, y \rangle$, and the storage function V satisfies $V(0) = 0$. △

In other words, a system Σ is passive if there exists a C° nonnegative function V: $X \to \boldsymbol{R}$, which satisfies $V(0) = 0$, such that

$$V(x) - V(x^\circ) \leq \int_0^t y^{\mathrm{T}}(s) u(s) ds. \tag{2.2}$$

Remark 2.5: Setting $u = 0$, we see from this definition that V is decreasing along any unforced trajectory of (2.1); it follows then that passive systems having a positive definite storage function V are Lyapunov stable. Reciprocally, we see also that V is decreasing along any trajectory of (2.1) consistent with the constraint $y = 0$. Since all such trajectories define what are called the *zero dynamics* of a system [1], we can deduce that passive systems having a positive definite storage function V have a Lyapunov stable zero dynamics. △

Sometimes, among the passive systems, it is convenient to identify those systems corresponding to the two limiting situations in which the dissipation inequality (2.2) becomes either a strict equality or (somewhat loosely speaking) a strict inequality. These two classes are characterized by the following definitions.

Definition 2.6: A passive system Σ with storage function V is said to be *lossless* if for all $u \in \mathscr{U}$, $x^\circ \in X$, $t \geq 0$

$$V(x) - V(x^\circ) = \int_0^t y^{\mathrm{T}}(s) u(s) ds. \quad \triangle$$

Definition 2.7: A passive system Σ with storage function V is said to be *strictly passive* if there exists a positive definite function $S: X \to \boldsymbol{R}$ such that for all $u \in \mathscr{U}$, $x^\circ \in X$, $t \geq 0$

$$V(x) - V(x^\circ) = \int_0^t y^{\mathrm{T}}(s) u(s) ds - \int_0^t S(x(s)) ds. \quad \triangle$$

Passive systems are related to *positive real* systems. The latter can be defined as follows.

Definition 2.8: A system Σ is said to be *positive real* if for all $u \in \mathscr{U}$, $t \geq 0$

$$0 \leq \int_0^t y^{\mathrm{T}}(s) u(s) ds$$

whenever $x(0) = 0$. △

Remark 2.9: In the case of a linear system, it is easily seen from Parseval's identity that the integral inequality for $t = \infty$ is equivalent to nonnegativity of the real part of the system transfer function on the imaginary axis. More generally, it follows from an integral transform argument that the integral inequality for all nonnegative t is equivalent to positive realness of the transfer function, i.e., that the transfer function be analytic and have nonnegative real part in the open right-half plane. △

The relation between passive and positive real systems depends on the property, of the state-space realization, of being reachable from the equilibrium point $x = 0$. We recall that a state x is *reachable* from 0 if there exists $t > 0$ and $u \in \mathscr{U}$ such that $x = \Phi(t, 0, u)$.

Proposition 2.10 [26]: A dissipative system Σ with supply rate $w = y^{\mathrm{T}} u$ is positive real if and only if its available storage satisfies $V_a(0) = 0$. A passive system is positive real. Conversely, if a system is positive real, its available storage is finite at each x which is reachable from the origin; as a consequence, a positive real system in which any state is reachable from the origin and in which V_a is C° is passive.

Proof: The first part of the statement is a straightforward consequence of the definition of available storage because $V_a(0) \geq 0$ and

$$V_a(0) = \sup_{\substack{x^\circ = 0 \\ u \in \mathscr{U} \\ t \geq 0}} \left\{ -\int_0^t y^{\mathrm{T}}(s) u(s) ds \right\}.$$

The second part of the statement is a straightforward consequence of the dissipation inequality (2.2), because V is nonnegative and vanishes at $x = 0$. The last part of the statement has been proven in [26] (see also [10]). △

We now turn to another fundamental property of passive systems which is one nonlinear enhancement of the ubiquitous Kalman–Yacubovitch–Popov lemma for positive real linear systems.

Definition 2.11: A system Σ has the KYP property if there exists a C^1 nonnegative function $V: X \to \boldsymbol{R}$, with $V(0) = 0$, such that

$$L_f V(x) \leq 0 \tag{2.3a}$$

$$L_g V(x) = h^{\mathrm{T}}(x) \tag{2.3b}$$

for each $x \in X$. △

The two relations (2.3) can be interpreted as the *infinitesimal* version of the dissipation inequality (2.2) for a passive system (although one could, as in the papers [11], [12], [13], view the dissipation inequality itself as another nonlinear version of the Kalman–Yacubovitch–Popov lemma). Concerning (2.3) it is possible to prove, as in [10], the following result.

Proposition 2.12 [10]: A system Σ which has the KYP property is passive, with storage function V. Conversely, a passive system having a C^1 storage function has the KYP property.

Proof: If Σ has the KYP property, then along any of its trajectories

$$\frac{dV(x(t))}{dt} = L_f V(x(t)) + L_g V(x(t)) u(t) \leq y^{\mathrm{T}}(t) u(t) \tag{2.4}$$

and integration form 0 to t yields the dissipation inequality (2.2) for the storage function V. Conversely, if Σ is passive with a C^1 storage function V, taking the derivative with respect to time of the left-hand side of the dissipation inequality yields (2.4), which clearly implies (2.3). △

Remark 2.13: Note that, in a lossless system with C^1 storage function V

$$L_f V(x) = 0$$

for all $x \in X$, and, in a strictly passive system with C^1

storage function V

$$L_f V(x) = -S(x)$$

for all $x \in X$, i.e., $L_f V$ is negative definite. Thus, in particular, if a system is strictly passive with a storage function which is positive definite, its equilibrium point $x = 0$ is asymptotically stable. △

Remark 2.14: Consider a system of the form (2.1a), i.e., with no specific output defined, and suppose there exists a C^1 nonnegative function $V: X \to \boldsymbol{R}$, with $V(0) = 0$, satisfying (2.3a), i.e., such that $L_f V(X) \leq 0$. Several authors (see, e.g., [7], [15], [16], [18]) have studied the problem of stabilizing such a system using a state feedback of the form

$$u = -\left[L_g V(x)\right]^{\mathrm{T}}.$$

In view of (2.3) and of Proposition 2.12, we observe that this control law can be interpreted as a unit gain negative output feedback

$$u = -y$$

imposed on the passive system defined choosing for (2.1a) the output map

$$y = \left[L_g V(x)\right]^{\mathrm{T}}.$$

We will return to this point in the next section. △

Remark 2.15: As pointed out verbally by J. Ball, G. Picci, and in a review by an anonymous referee, the condition that a system Σ be positive real is equivalent to the condition that an associated system Σ' has a finite L_2 gain. More precisely, defining a new input w and a new output z via

$$u = \gamma w + z$$
$$y = \gamma w - z$$

the inequality introduced in Definition 2.8 takes the form

$$\int_0^t z^{\mathrm{T}}(s) z(s)\, ds \leq \gamma^2 \int_0^t w^{\mathrm{T}}(s) w(s)\, ds. \qquad \triangle$$

III. Stabilization by Output Feedback.

In this section, we revisit a certain number of known results about the possibility of asymptotically stabilizing a nonlinear passive system by means of memoryless output feedback. The asymptotic stability of interconnected passive systems has been studied in depth in the literature by several authors, either from an operator theoretic point of view (as in [6], [20]–[23], [25], [28]) or in terms of the corresponding state-space descriptions (as in [10]–[13], [26], [27]). In particular, Hill and Moylan [10]–[13] have developed a synthesis of the techniques from the theory of passive systems and the Lyapunov stability theory which yields a number of important stability results under suitable observability hypotheses.

In the first part of this section, we will show how the observability condition used by Hill–Moylan can, in fact, be slightly weakened and brought to a form, that we call detectability, which is particularly suited to the analysis that will be presented in Sections IV and V. In particular, we will derive a direct criterion for detectability for a passive system, stated in terms of Lie brackets of the vector fields which characterizes the input-state description (2.1a), and is reminiscent of the well-known rank conditions for accessibility. This will enable us to show, in the second part of the section, that certain stabilization laws independently proposed in the literature on geometric nonlinear control, and some generalizations thereof, can all be derived from a basic stabilizability property of passive systems.

We first recall two basic definitions about observability and detectability.

Definition 3.1: A system Σ is *locally zero-state detectable* if there exists a neighborhood U of 0 such that, for all $x \in U$

$$h(\Phi(t, x, 0)) = 0 \quad \text{for all } t \geq 0 \Rightarrow \lim_{t \to \infty} \Phi(t, x, 0) = 0.$$

If $U = X$, the system is *zero-state detectable.* A system Σ is *locally zero-state observable* if there exists a neighborhood U of 0 such that, for all $x \in U$

$$h(\Phi(t, x, 0)) = 0 \quad \text{for all } t \geq 0 \Rightarrow x = 0.$$

If $U = X$, the system *zero-state observable.* △

These two definitions are natural extensions of well established concepts from linear system theory. Note however that in some of the literature on passive systems, the term *detectability* is used to mean what here is defined as *observability* (see, e.g., [10]).

The following statement, whose proof is a natural adaptation of the proof of LaSalle's invariance principle, describes a basic stabilizability property of passive systems. For more general informations about stability of interconnected passive systems, we refer to the papers [10]–[13] by Hill and Moylan. For convenience, we recall that a nonnegative function $V: X \to \boldsymbol{R}$ is said to be *proper* if for each $a > 0$, the set $V^{-1}([0, a]) = \{x \in X : 0 \leq V(x) \leq a\}$ is compact.

Theorem 3.2: Suppose Σ is passive with a storage function V which is positive definite. Suppose Σ is locally zero-state detectable. Let $\phi: Y \to U$ be any smooth function such that $\phi(0) = 0$ and $y^{\mathrm{T}}\phi(y) > 0$ for each nonzero y. The control law

$$u = -\phi(y) \tag{3.1}$$

asymptotically stabilizes the equilibrium $x = 0$. If Σ is zero-state detectable and V is proper, the control law (3.1) globally asymptotically stabilizes the equilibrium $x = 0$.

Proof: Along any trajectory of the closed-loop system (2.1)–(3.1), the dissipation inequality yields

$$V(x(t)) - V(x(0)) \leq -\int_0^t y^{\mathrm{T}}(s)\phi(y(s))\, ds \leq 0.$$

Thus, $V(x(t))$ is nonincreasing along the trajectories of the closed-loop system. Since V is positive definite, for any sufficiently small $a > 0$, the set $V^{-1}([0, a])$ is compact, and therefore the equilibrium $x = 0$ of the closed-loop is Lyapunov stable. Choose a sufficiently small initial condition x°, let $x^\circ(t)$ denote the corresponding trajectory and let γ°

denote its ω-limit set (which is nonempty, compact, and invariant). Since $\lim_{t\to\infty} V(x(t)) = a_0 \geq 0$, by continuity of V, $V(x) = a_0$ at each point x of γ°. Let $\bar{x}$ be a point of γ° and $\bar{x}(t)$ the corresponding trajectory. Since $\bar{x}(t) \in \gamma^\circ$, then $V(\bar{x}(t)) = a_0$ for all $t \geq 0$ and

$$0 = V(\bar{x}(t)) - V(\bar{x}) \leq -\int_0^t y^{\mathrm{T}}(s)\phi(y(s))\,ds \leq 0$$

implies $y(t) = 0$ for all $t \geq 0$. By detectability (observe that, since $u(t) = -\phi(y(t)) = 0$, closed-loop trajectories coincide with open-loop trajectories) $\lim_{t\to\infty} \bar{x}(t) = 0$ and therefore $a_0 = 0$. Thus, $\lim_{t\to\infty} V(x^\circ(t)) = 0$, i.e., $\lim_{t\to\infty} x^\circ(t) = 0$. This proves local asymptotic stability of the equilibrium $x = 0$. If V is proper, then $x = 0$ is globally asymptotically stable. △

The previous theorem shows that any passive system having a *positive definite* storage function V, if *zero-state detectable*, is (globally) asymptotically stabilized by *pure gain* output feedback. We will now describe how these assumptions can be tested and will use the conditions thus derived in order to state different criteria for stabilizability. For simplicity, we discuss only the conditions for global asymptotic stabilization. We first recall a result of Hill–Moylan ([10], Lemma 1) showing how the positive definiteness of V is implied by the property of zero-state observability.

Proposition 3.3 [10]: Suppose Σ is passive with storage function V. Suppose Σ is zero-state observable. Then V is positive definite.

Proof: By Proposition 2.3, the available storage is finite. Moreover, $V_a(0) = 0$ (because $V(0) = 0$). By definition

$$V_a(x) = \sup_{\substack{x^\circ = x\\ u\in\mathcal{U}\\ t\geq 0}} \left\{-\int_0^t y^{\mathrm{T}}(s)u(s)\,ds\right\} \geq \sup_{\substack{x^\circ = x\\ t\geq 0}} \left\{\int_0^t y^{\mathrm{T}}(s)y(s)\,ds\right\}.$$

If $V_a(x) = 0$, then necessarily $y(t) = 0$ and this, by zero state observability, yields $x = 0$. Thus V_a vanishes only at $x = 0$ and so does V, because $V(x) \geq V_a(x)$. △

The next result, which is slightly more subtle, describes conditions which imply zero-state detectability. The result itself is, to the best of our knowledge, a new result, although its proof is substantially based on a clever argument proposed by Lee–Arapostatis ([18, proof of Theorem 1]). In order to describe this result we need some preliminary material. With the vector fields $f, g_1, \cdots, g_m$ which characterize (2.1a) we associate the distribution

$$\mathfrak{D} = \operatorname{span}\{\operatorname{ad}_f^k g_i : 0 \leq k \leq n-1, 1 \leq i \leq m\}.$$

Moreover, we recall that (cf. proof of Theorem 3.2) for a passive system having a C^1 storage function V which is positive definite and proper, for any initial condition $x^\circ \in X$, the trajectory $\Phi(\cdot, X^\circ, 0)$ is bounded, and the associated limit set is nonempty and compact. Set

$$\Omega = \bigcup_{x^\circ\in X} (\omega\text{-limit set of } \Phi(\cdot, x^\circ, 0)).$$

We can show that the objects thus introduced are useful in testing the zero-state detectability and/or observability of a passive system.

Proposition 3.4: Suppose Σ is passive with a proper C^r, $r \geq 1$, storage function V. Let S denote the set

$$S = \{x \in X : L_f^m L_\tau V(x) = 0,\ \text{for all } \tau \in \mathfrak{D}, \text{all } 0 \leq m < r\}.$$

If $S \cap \Omega = \{0\}$ and V is positive definite, then Σ is zero-state detectable. If $S = \{0\}$ and Σ is lossless, then Σ is zero-state observable.

Proof: Let $x(t)$ be a unforced trajectory yielding $y(t) = 0$ for all $t \geq 0$ and let γ° denote its ω-limit set. Let $\bar{x}$ be a point of γ° and $\bar{x}(t)$ the corresponding trajectory. From the proof of Theorem 3.2 it is known that $V(x)$ is constant on γ°. Therefore $V(\bar{x}(t))$ is constant and

$$\frac{dV(\bar{x}(t))}{dt} = L_f V(\bar{x}(t)) = 0,$$

i.e., $L_f V$ is maximal on $\bar{x}(t)$. Moreover, since $L_g V(x(t)) = 0$ (see 2.3b)), we see that $L_g V(x)$ vanishes on γ°, and in particular $L_g V(\bar{x}(t)) = 0$. From these conditions, continuing as in [18, Proof of Theorem 1] one deduces that

$$L_f^m L_\tau V(\bar{x}) = 0, \qquad \text{for all } \tau \in \mathfrak{D}, \text{all } 0 \leq m < r. \tag{3.2}$$

In fact, for $\tau = [f, g]$,

$$\begin{aligned} L_\tau V(\bar{x}(t)) &= L_f L_g V(\bar{x}(t)) - L_g L_f V(\bar{x}(t)) \\ &= \frac{dL_g V(\bar{x}(t))}{dt} - \frac{\partial L_f V}{\partial x} g(\bar{x}(t)) = 0 \end{aligned}$$

because $\partial L_f V/\partial x$ vanishes on $\bar{x}(t)$, where $L_f V$ is maximal. An easy recursion and the fact that γ° is invariant under f yield (3.2). Since $\bar{x} \in \Omega$, the condition $S \cap \Omega = \{0\}$ implies $\bar{x} = 0$. Therefore $V(\bar{x}) = 0$ and this implies (as in Theorem 3.2) $\lim_{t\to\infty} x(t) = 0$, thus completing the proof of zero-state detectability. If Σ is lossless, similar arguments prove that (3.2) holds at any $\bar{x} \in X$, provided $h(\bar{x}(t)) = 0$. Thus $S = \{0\}$ implies zero-state observability. △

Using either one of the conditions described in Propositions 3.3 and 3.4 in order to check the assumptions required by the basic stabilization strategy expressed by Theorem 3.2, it is possible to recover a number of stabilization results independently proposed in the literature by various authors. This shows that a number of apparently independent stabilization schemes reduce, in fact, to the one of a passive system subject to pure gain output feedback. The first result, due to Hill–Moylan ([10, Theorem 2]), is a straightforward combination of Theorem 3.2 with Proposition 3.3.

Corollary 3.5: Suppose Σ is passive with proper storage function V. Suppose Σ is zero-state observable. For each $k > 0$ control law $u = -ky$ globally asymptotically stabilizes the equilibrium $x = 0$.

The second result can be deduced from a combination of Theorem 3.2 with the condition for zero-state detectability expressed by Proposition 3.4.

Corollary 3.6: Suppose Σ is passive with a C^r, $r \geq 1$, and proper storage function V. If $S \cap \Omega = \{0\}$ and V is positive definite, for each $k > 0$ the control law $u = -ky$ globally asymptotically stabilizes the equilibrium $x = 0$.

The third result can be deduced from a combination of Theorem 3.2 with both Propositions 3.4 (the condition for zero-state observability) and Proposition 3.3

Corollary 3.7: Suppose Σ is lossless with a C^r, $r \geq 1$, and proper storage function V. If $S = \{0\}$, for each $k > 0$ the control law $u = -ky$ globally asymptotically stabilizes the equilibrium $x = 0$.

Remark 3.8: The results expressed by Corollaries 3.6 and 3.7 can be interpreted as generalizations of rather well known results [7], [15], [16], [18] on global feedback stabilization of systems:

$$\dot{x} = f(x) + g(x)u \tag{3.3}$$

obtained using methods from nonlinear geometric control theory. In [7] (which generalizes the results of Jurdjevic and Quinn [15]), Gauthier and Bornard assume that there exists a function $V(x)$ with no critical point other than $x = 0$ and such that

$$L_f V(x) = 0 \tag{3.4}$$

and prove that, if $\mathfrak{D}$ has dimension n at each $x \neq 0$, the control law

$$u = -(L_g V(x))^{\mathrm{T}} \tag{3.5}$$

globally asymptotically stabilizes the system. As discussed in Remark 2.14, the control law (3.5) is a direct output feedback on a system of the form (3.3) having an output $y = (L_g V(x))^{\mathrm{T}}$, which is lossless by (3.4) and Proposition 2.12. Clearly, the hypotheses of [7] imply $S = \{0\}$, and therefore [7, Theorem 1] is a particular case of Corollary 3.7. In [18], Lee and Arapostathis derive a more general result, which also includes a result of [16], proving that the feedback law (3.5) globally asymptotically stabilizes the system under the weaker hypothesis that $L_f V(x) \leq 0$ and $S = \{0\}$. Therefore, again viewing the state feedback law (3.5) as a direct output feedback on a passive system, one can interpret Corollary 3.6 as a slight extension of [18, Theorem 1]. △

IV. Feedback Equivalence to a Passive System

In this section, we discuss conditions under which a given system is feedback equivalent to a passive system with positive definite storage function V. Since, as we shall see in a moment, a role of major importance is played by property—for the system—of being minimum phase, we briefly recall how minimum phase nonlinear systems are characterized. We assume that the reader is familiar with the concepts of *relative degree* (see [1] or [14] for details) and *normal form*. In particular, we recall that a system of the form (2.1) is said to have relative degree $\{1, \cdots, 1\}$ at $x = 0$ if the matrix $L_g h(0)$ is nonsingular. If this is the case and if the distribution spanned by the vector fields $g_1(x), \cdots, g_m(x)$ is involutive, it is possible to find $n - m$ real-valued functions $z_1(x), \cdots, z_{n-m}(x)$, locally defined near $x = 0$ and vanishing at $x = 0$, which, together with the m components of the output map $y = h(x)$, qualify as a new set of local coordinates. In the new coordinates (z, y), the system is represented by equations having the following structure (*normal form*)

$$\dot{z} = q(z, y) \tag{4.1a}$$

$$\dot{y} = b(z, y) + a(z, y)u \tag{4.1b}$$

where the matrix $a(z, y)$ is nonsingular for all (z, y) near $(0, 0)$.

The *zero dynamics* of a system (see [1], [2], [14]) describe those internal dynamics which are consistent with the external constraint $y = 0$. If a system has relative degree $\{1, \cdots, 1\}$ at $x = 0$, its zero dynamics locally exist in a neighborhood U of $x = 0$, evolve on the smooth $(n - m)$-dimensional submanifold

$$Z^* = \{x \in U : h(x) = 0\}$$

(the *zero dynamics manifold*) and are described by a differential equation of the form

$$\dot{x} = f^*(x) \qquad x \in Z^*$$

in which $f^*(x)$ (the *zero dynamics vector field*) denotes the restriction to Z^* of the vector field

$$\tilde{f}(x) = f(x) + g(x)u^*(x) \tag{4.2}$$

with

$$u^*(x) = -[L_g h(x)]^{-1} L_f h(x).$$

In the normal form (4.1) the zero dynamics are characterized by the equation

$$\dot{z} = q(z, 0).$$

In view of this, we shall sometimes denote—with a minor abuse of notation—$q(z, 0)$ by $f^*(z)$ and express $q(z, y)$ in the form

$$q(z, y) = f^*(z) + p(z, y)y$$

where $p(z, y)$ is a smooth function.

In [4], necessary and sufficient conditions for the existence of a *globally defined* normal form of the type (4.1) have been investigated. In addition to the nonsingularity of the matrix $L_g h(x)$, these conditions require further properties on set of m vector fields $\tilde{g}_1(x), \cdots, \tilde{g}_m(x)$ defined by

$$[\tilde{g}_1(x) \cdots \tilde{g}_m(x)] = g(x)[L_g h(x)]^{-1}. \tag{4.3}$$

More precisely, there exists a globally defined diffeomorphism which transforms the system (2.1) into a system having the normal form (4.1) if and only if:

H1: the matrix $L_g h(x)$ is nonsingular for each $x \in X$,
H2: the vector fields $\tilde{g}_1(x), \cdots, \tilde{g}_m(x)$ are complete,
H3: the vector fields $\tilde{g}_1(x), \cdots, \tilde{g}_m(x)$ commute.

If this is the case, then *globally defined* zero dynamics exist for the system. Note that the condition H3 is equivalent to the condition that the distribution spanned by $g_1(x), \cdots, g_m(x)$ is involutive.

A system whose zero dynamics are asymptotically stable has been called a *minimum* phase system (see [1]–[3]). In the following definition, we specialize this concept in a more detailed manner.

Definition 4.1: Suppose $L_g h(0)$ is nonsingular. Then Σ is said to be:

i) *minimum phase* if $z = 0$ is an asymptotically stable equilibrium of $f^*(z)$,

ii) *weakly minimum phase* if there exists a C^r, $r \geq 2$, function $W^*(z)$, locally defined near $z = 0$ with $W^*(0) = 0$, which is positive definite and such that $L_{f*}V(z) \leq 0$ for all z near $z = 0$.

Suppose H1, H2, and H3 hold. Then Σ is said to be:

iii) *globally minimum phase* if $z = 0$ is a globally asymptotically stable equilibrium of $f^*(z)$,

iv) *globally weakly minimum phase* if there exists a C^r, $r \geq 2$, function $W^*(z)$, defined for all z with $W^*(0) = 0$, which is positive definite and proper such that $L_{f*}V(z) \leq 0$ for all z. △

Remark 4.2: Note that in a weakly minimum phase system the equilibrium $z = 0$ of the zero dynamics vector field is stable in the sense of Lyapunov (but possibly not asymptotically stable). However, if the equilibrium $z = 0$ of the zero dynamics vector field is stable in the sense of Lyapunov, the system needs not be weakly minimum phase in the sense of Definition 4.1, ii). This reflects the fact that stability in the sense of Lyapunov does not imply, in general, the existence of a t-independent positive definite function $W^*(z)$ whose derivative (along the trajectories of the system) is negative semidefinite (see, e.g., [7, p.228]). △

We proceed now to illustrate how the concepts of relative degree and zero dynamics arise naturally in the study of passive systems, playing in fact an important role. We begin by analyzing the relative degree of a passive system. In what follows, for convenience, we will say that a point x° is a *regular* point for a system Σ of the form (2.1) if rank $\{L_g h(x)\}$ is constant in a neighborhood of x°. We also assume throughout the section that rank $\{g(0)\}$ = rank $\{dh(0)\} = m$.

Theorem 4.3: Suppose Σ is passive with a C^2 storage function V which is positive definite. Suppose $x = 0$ is a regular point for Σ. Then $L_g h(0)$ is nonsingular and Σ has relative degree $\{1, \cdots, 1\}$ at $x = 0$.

Proof: If $L_g h(0)$ is singular, there exists a smooth $\boldsymbol{R}^m$-vector-valued function u, defined in a neighborhood U of $x = 0$, such that

$$L_g h(x) u(x) = 0$$

$$\gamma(x) = g(x) u(x) \neq 0$$

for each $x \in U$. Note also that, because of (2.3b)

$$\begin{aligned} L^2\gamma V(x) &= L_\gamma[L_g V(x) u(x)] = L_\gamma[u^{\mathrm{T}}(x) h(x)] \\ &= [L_\gamma u^{\mathrm{T}}(x)] h(x) + u^{\mathrm{T}}(x) L_\gamma h(x) \\ &= v^{\mathrm{T}}(x) h(x) \end{aligned}$$

(where $v^{\mathrm{T}}(x) = L\gamma u^{\mathrm{T}}(x)$).

Let $\Phi_t^\gamma(x)$ denote the flow of the vector field γ, set $f(t) = V(\Phi_g^\gamma(0))$ and consider the expansion

$$f(t) = f(0) + f^{(1)}(0)t + f^{(2)}(s)\frac{1}{2}t^2$$

where $0 \leq s \leq t$. Since $f^{(k)}(t) = L_\gamma^k V(\Phi_t^\gamma(0))$, for $k = 0, 1, 2$, we have

$$\begin{aligned} V(\Phi_t^\gamma(0)) &= V(0) + L_\gamma V(0)t + L_\gamma^2 V(\Phi_s^\gamma(0))\frac{1}{2}t^2 \\ &= v^{\mathrm{T}}(\Phi_s^\gamma(0)) h(\Phi_s^\gamma(0))\frac{1}{2}t^2 \end{aligned}$$

because $V(0) = 0$, $L_\gamma V(0) = u^{\mathrm{T}}(0)h(0) = 0$, and $L_\gamma^2 V(x) = v^T(x)h(x))$. By definition $dh(x)\gamma(x) = 0$, i.e., the vector field γ is tangent to the level sets of h (which are locally smooth submanifolds, by assumption). Therefore $h(\Phi_s^\gamma(0)) = h(0)$, i.e.,

$$V(\Phi_t^\gamma(0)) = 0$$

for all t. This in turn implies $\Phi_t^\gamma(0) = 0$, because $x = 0$ is an isolated zero of V, which is a contradiction, because $\gamma(0) \neq 0$. △

In the case $m = 1$, the previous result yields the following interesting consequence.

Corollary 4.4: Suppose Σ is passive with a C^2 storage function V which is positive definite. Suppose $m = 1$. Then, in any neighborhood of the point $x = 0$ there is a point where $L_g h$ is nonzero.

Proof: If $x = 0$ is a regular point, then Theorem 4.3 applies and $L_g h(0)$ is nonzero. If $x = 0$ is not a regular point, then $L_g h$ is necessarily zero at $x = 0$ but not identically zero in a neighborhood of $x = 0$. △

The following proposition provides an independent sufficient condition, stated in terms of the storage function, for a passive system to have relative degree $\{1, \cdots, 1\}$ at $x = 0$.

Proposition 4.5: Suppose Σ is passive with a C^2 storage function V which is positive definite. Suppose V is nondegenerate at $x = 0$. Then $L_g h(0)$ is nonsingular and Σ has relative degree $\{1, \cdots, 1\}$ at $x = 0$.

Proof: By contradiction, suppose there exists a vector u such that $L_g h(0)u = 0$ and set $\gamma = g(0)u$. It follows that

$$\begin{aligned} 0 &= u^{\mathrm{T}} L_g h(0) u = L_\gamma^2 V(0) \\ &= \gamma^{\mathrm{T}} H \gamma + \left(\frac{\partial V}{\partial x}\right)_0 (L_\gamma g(0)) u = \gamma^{\mathrm{T}} H \gamma \end{aligned}$$

where H is the Hessian matrix of V evaluated at $x = 0$. The latter is positive definite by hypothesis, and therefore u is necessarily the zero vector. △

The next result characterizes the asymptotic properties of the zero dynamics of a passive system.

Theorem 4.6: Suppose Σ is passive with a C^2 storage function V which is positive definite. Suppose that either $x = 0$ is a point of regularity for Σ or that V is nondegenerate. Then the zero dynamics of Σ locally exist at $x = 0$ and Σ is weakly minimum phase.

Proof: By theorem 4.3, Σ has relative degree $\{1, \cdots, 1\}$ at $x = 0$ and its zero dynamics indeed locally exist at $x = 0$. The function $W^*: Z^* \to \mathbf{R}$ defined by

$$W^* = V|_{Z^*}$$

is positive definite. Moreover, its derivative along trajectories of the zero dynamics vector field is negative semidefinite in a neighborhood of the origin. In fact (cf. (2.3))

$$0 \geq L_f V(x) = L_{f^*}V(x) - L_g V(x) u^*(x) = L_{f^*}V(x) - h^{\mathrm{T}}(x) u^*(x) = L_{f^*}V(x)$$

because $h(x) = 0$ along any trajectory of the zero dynamics. △

Theorems 4.3 and 4.6 show, in essence, that *any passive system* with a positive definite storage function, under mild regularity assumptions, necessarily has *relative degree* $\{1, \cdots, 1\}$ at $x = 0$ and is *weakly minimum phase*. The next step of our investigation is to show that exactly these two conditions characterize the *equivalence*, via *state feedback*, to a passive system. We consider here regular static (i.e., memoryless) state feedback, i.e., feedback of the form

$$u = \alpha(x) + \beta(x) v$$

where $\alpha(x)$ and $\beta(x)$ are smooth functions defined either locally near $x = 0$ or globally, and $\beta(x)$ is invertible for all x. The necessity of these conditions follows immediately from the fact that both relative degree and zero dynamics are invariant under feedback [4], [14]. The sufficiency is less straightforward (but still easy) and is illustrated in the proof of the following statement.

Theorem 4.7: Suppose $x = 0$ is a regular point for Σ. Then Σ is locally feedback equivalent to a passive system with a C^2 storage function V, which is positive definite, if and only if Σ has relative degree $\{1, \cdots, 1\}$ at $x = 0$ and is weakly minimum phase.

Proof: Choosing as new state variables $y = h(x)$ and any complementary set $\eta = \phi(x)$, the system is represented by equations of the form

$$\dot{\eta} = c(\eta, y) + d(\eta, y) u$$
$$\dot{y} = b(\eta, y) + a(\eta, y) u$$

where $a(\eta, y)$ is nonsingular for all (η, y) near $(0, 0)$. Then, imposing the feedback law

$$u = a(\eta, y)^{-1}[-b(\eta, y) + v]$$

changes Σ into a system described by equations of the form

$$\dot{\eta} = \theta(\eta, y) + \gamma(\eta, y) v$$
$$\dot{y} = v.$$

After the additional change of variables

$$z = \eta - \gamma(\eta, 0) y$$

the system becomes

$$\dot{z} = f^*(z) + p(z, y) y + \left(\sum_{i=1}^{m} q_i(z, y) y_i \right) v \quad (4.4a)$$

$$\dot{y} = v \quad (4.4b)$$

where $p(z, y)$ and the $q_i(z, y)$'s are suitable matrices of appropriate dimensions. Recall now that if Σ is weakly minimum phase, there exists a C^2 positive definite function $W^*(z)$, with $W^*(0) = 0$, such that $L_{f^*}W^*(z) \leq 0$ for each $z \neq 0$. Define the matrix

$$M(z, y) = \begin{bmatrix} L_{q_1(z, y)} W^*(z) \\ \cdots \\ L_{q_m(z, y)} W^*(z) \end{bmatrix}$$

and note that, by construction, $M(0, y) = 0$ (because $z = 0$ is a minimum of $W^*(z)$). Therefore the feedback law

$$v = [I + M(z, y)]^{-1}\left[-(L_{p(z, y)} W^*(z))^{\mathrm{T}} + w \right] \quad (4.5)$$

is well defined in a neighborhood of $(0, 0)$.

Consider now the closed-loop system (4.4)–(4.5), which has a form

$$\begin{bmatrix} \dot{z} \\ \dot{y} \end{bmatrix} = \bar{f}(z, y) + \bar{g}(z, y) w$$

together with the positive definite and C^2 function

$$V(z, y) = W^*(z) + \frac{1}{2} y^{\mathrm{T}} y.$$

A straightforward calculation shows that

$$\begin{aligned} L_{\bar{f}} V(z, y) + L_{\bar{g}} V(z, y) w &= \dot{V}(z, y) = L_{f^*} W^*(z) \\ &\quad + y^{\mathrm{T}} \left[(L_{p(z, y)} W^*(z))^{\mathrm{T}} + M(z, y) v \right] \\ &\quad + y^{\mathrm{T}} v = L_{f^*} W^*(z) + y^{\mathrm{T}} w. \end{aligned}$$

Therefore

$$L_{\bar{f}} V(z, y) = L_{f^*} W^*(z) \leq 0 \qquad (L_{\bar{g}} V(z, y))^{\mathrm{T}} = y.$$

This, in view of Proposition 2.12, completes the proof. △

Remark 4.8: In a linear system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

with $\operatorname{rank}\{B\} = m$, $x = 0$ is always a regular point and a normal form-whenever it exists—is globally defined. Thus, from the previous result we immediately obtain that any linear system is feedback equivalent to a passive linear system with a storage function $V(x) = x^{\mathrm{T}} Q x$, which is positive definite, if and only if CB is nonsingular and the system is weakly minimum phase. Since any controllable linear system is passive, with a storage function $V(x) = x^{\mathrm{T}} Q x$ which is positive definite, if and only if it is *positive real* (see, e.g., [27]), we can also deduce from Theorem 4.7 the result, recently proven by Saberi, Kokotovic, and Sussmann [24], that any controllable linear system is feedback equivalent to a positive real system if and only if CB is nonsingular and the system is weakly minimum phase. △

Remark 4.9: Note that the proof of the previous theorem greatly simplifies whenever a local normal form of the type (4.1) exists (which is the case whenever the distribution spanned by the vector fields $g_1(x), \cdots, g_m(x)$ is involutive).

In fact, in the normal form (4.1) there is no direct influence of the input on the variable z. As a consequence, (4.4a) reduces to

$$\dot{z} = f^*(z) + p(z, y)y$$

and the feedback law (4.5) reduces to

$$v = -\left(L_{p(z,y)}W^*(z)\right)^{\mathrm{T}} + w. \qquad \triangle$$

A global version of Theorem 4.5 indeed exists if the system in question has a global form. Using the assumptions H1–H3 we have, in fact, the following result where, for more generality, also the case of global feedback equivalence to a strictly passive system is considered.

Theorem 4.10: Assume H1–H3. Then Σ is globally feedback equivalent to a passive (respectively, strictly passive) system with a C^2 storage function V, which is positive definite, if and only if Σ is globally weakly minimum phase (respectively, globally minimum phase).

So far, we have investigated the feedback equivalence of a given system to a passive system with positive definite storage function V. In the next statement, we analyze the particular configuration in which the system in question can be expressed in the form

$$\dot{\zeta} = f_0(\zeta) + f_1(\zeta, y)y \qquad (4.6a)$$
$$\dot{x} = f(x) + g(x)u \qquad (4.6b)$$
$$y = h(x) \qquad (4.6c)$$

which we assume to be globally valid (of course, corresponding local results also hold). The analysis of configurations of this type was considered in a number of previous papers (see, e.g., [3], [17], [24]). In view of the particular structure of (4.6), we will call

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

the *driving system*, while

$$\dot{\zeta} = f_0(\zeta) + f_1(\zeta, y)y$$

will be called the *driven system*.

We now examine conditions under which this type of system is feedback equivalent to a passive system; in the next section we will use these conditions, together with the results established in Section III, to show how systems of this type can be globally asymptotically stabilized via smooth feedback.

First, note that if the point $(\zeta, x) = (0,0)$ were a point of regularity for the full system (4.6), then its local feedback equivalence to a passive system would be of course determined by the conditions described in Theorem 4.7, namely the properties of having relative degree $\{1,\cdots,1\}$ at $(\zeta, x) = (0,0)$ and of being weakly minimum phase. Note also that, in view of the special structure of (4.6) the point $(\zeta, x) = (0,0)$ is a point of regularity for the full system if and only if the point $x = 0$ is a point of regularity for the driving system and that, in particular, the full system has relative degree $\{1,\cdots,1\}$ at $(\zeta, x) = (0,0)$ if and only if the driving system has relative degree $\{1,\cdots,1\}$ at $x = 0$. Finally, note that, in this case (that is, if $L_g h(0)$ is nonsingular) the zero dynamics of the full system have the form

$$\dot{\zeta} = f_0(\zeta) \qquad (4.7a)$$
$$\dot{z} = f^*(z) \qquad (4.7b)$$

where $f^*(z)$ is exactly the zero dynamics vector field of the driving system. Thus, the full system is weakly minimum phase if and only if the driving system is, and there exists a positive definite function $U(\zeta)$, locally defined near $\zeta = 0$ with $U(\zeta) = 0$, such that $L_{f_0}U(\zeta) \le 0$ for all ζ. Similar considerations can be repeated in a global setting, and we can therefore deduce, as an immediate application of our previous discussion, the following result.

Corollary 4.11: Suppose the triplet $\{f, g, h\}$ satisfies the assumptions H1–H3 (or, what is the same, suppose a normal form of the type (4.1) globally exists for the driving system of (4.6)). Then

i) the full system (4.6) is feedback equivalent to a passive system with a C^2 storage function V, which is positive definite, if and only if the driving system is weakly minimum phase and there exists a C^2 positive definite function $U(\zeta)$, defined for all $\zeta = 0$ with $U(\zeta) = 0$, such that $L_{f_0}U(\zeta) \le 0$ for all ζ;

ii) the full system (4.6) is feedback equivalent to a strictly passive system with a C^2 storage function V, which is positive definite, if and only if the driving system is globally minimum phase and

$$\dot{\zeta} = f_0(\zeta)$$

is globally asymptotically stable.

Remark 4.12: In the paper [17], the problem of stabilizing the configuration (4.6) when the driving system is linear and controllable was considered, following earlier work on stabilization of minimum phase nonlinear systems in normal form [3]. To this end, the auxiliary problem of characterizing those linear systems which can be rendered positive real via state feedback was posed. As we pointed out in Remark 4.6, the solution to this problem can be deduced from the specialization of our results on feedback equivalence to a passive system for *nonlinear* systems; namely, the system must have an invertible "high-frequency gain" matrix CB and must be weakly minimum phase. In view of this, a system

$$\dot{\zeta} = f_0(\zeta) + f_1(\zeta, y)y$$
$$\dot{x} = Ax + Bu$$
$$y = Cx$$

in which the unforced driven system is globally asymptotically stable and the driving system is feedback equivalent to a positive-real system, is indeed a globally weakly minimum phase system having relative degree $\{1,\cdots,1\}$ (as can be seen from the global decomposition (4.7) of its zero dynamics). In view of Corollary 4.11 the whole system is itself feedback equivalent to a passive system having a positive definite storage function and stabilization results such as those outlined in Section III apply (see Section V). $\triangle$

In the next statement, we show how feedback equivalence of (4.6) to a passive system can be determined *without assuming the existence of a normal form* for the driving system.

Theorem 4.13: Consider the system (4.6). Suppose

$$\dot{\zeta} = f_0(\zeta) \tag{4.8}$$

is globally asymptotically stable and $\{f, g, h\}$ is (strictly) passive with a C^r, $r \geq 1$, storage function V, which is positive definite. The system is feedback equivalent to a (strictly) passive system with a C^r storage function which is positive definite.

Proof: Let $U(\zeta)$ be a C^r Lyapunov function for (4.8) and consider

$$W(\zeta, x) = U(\zeta) + V(x).$$

The full system (4.6) has the form

$$\begin{aligned}\dot{\xi} &= F(\xi) + G(\xi)u\\ y &= H(\xi)\end{aligned}$$

with $\xi = (\zeta, x)$

$$\begin{aligned}F(\xi) &= \operatorname{col}(f_0(\zeta) + f_1(\zeta, h(x))h(x), f(x))\\ G(\xi) &= \operatorname{col}(0, g(x))\\ H(\xi) &= h(x).\end{aligned}$$

By (2.3b)

$$L_G W(\xi) = L_g V(x) = h^{\mathrm{T}}(x) = H^{\mathrm{T}}(\xi)$$

and

$$L_{F+G\alpha}W(\xi) = L_{f_0(\zeta)}U(\zeta) + L_{f_1(\zeta, h(x))}U(\zeta)h(x) + L_f V(x) + h^{\mathrm{T}}(x)\alpha.$$

Choosing

$$\alpha^{\mathrm{T}}(\xi) = -L_{f_1(\zeta, h(x))}U(\zeta)$$

yields, by (2.3a)

$$L_{F+G\alpha}W(\xi) = L_{f_0(\zeta)}U(\zeta) + L_f V(x) \leq 0$$

with a strict inequality (for all nonzero ξ) if the driving system is strictly passive. Therefore, by Proposition 2.12, we conclude that the full system is rendered passive by the feedback

$$u = \alpha(\xi) + v. \qquad \triangle$$

We conclude this section with a solution to a feedback equivalence problem of a different type: given a control system

$$\dot{x} = f(x) + g(x)u \tag{4.9}$$

when does there exist a smooth state feedback law $\alpha(x)$ and an output map $h(x)$ such that the resulting closed-loop input–output system is strictly passive, with a positive definite, proper storage function? Clearly, a necessary condition is that (4.9) be globally asymptotically stabilizable. The following proposition shows that actually this condition is also sufficient.

Proposition 4.14: A necessary and sufficient condition for the existence of a smooth feedback law $\alpha(x)$ and an output map $h(x)$ such that

$$\begin{aligned}\dot{x} &= f(x) + g(x)\alpha(x) + g(x)u\\ y &= h(x)\end{aligned}$$

is strictly passive, with a positive definite and proper storage function, is that (4.9) be globally asymptotically stabilizable by smooth state feedback.

Proof: Suppose (4.9) is globally asymptotically stabilizable, via $u = \alpha(x)$, and let V be a positive definite proper Lyapunov function for the corresponding closed-loop system. Setting $h^{\mathrm{T}}(x) = L_g V(x)$ yields a strictly passive system by Proposition 2.12. $\triangle$

V. Global Stabilization of Weakly Minimum Phase Systems

In this section, we apply some of the results illustrated so far to the problem of deriving globally asymptotically stabilizing feedback laws for certain classes of nonlinear systems. In particular, we give a fairly general theorem which, as we describe in the following, incorporates and extends a number of interesting results which recently appeared in the literature.

Theorem 5.1: Consider a system Σ described by

$$\dot{\zeta} = f_0(\zeta) + f_1(\zeta, y)y \tag{5.1a}$$

$$\dot{x} = f(x) + g(x)u \tag{5.1b}$$

$$y = h(x). \tag{5.1c}$$

Suppose

$$\dot{\zeta} = f_0(\zeta) \tag{5.2}$$

is globally asymptotically stable. Suppose $\{f, g, h\}$ is passive with a C^r, $r \geq 1$, storage function V, which is positive definite and proper, and suppose $S = \{0\}$ with S defined as in Section III. Then Σ is globally asymptotically stabilizable by smooth state feedback.

Proof: By Theorem 4.13, Σ is feedback equivalent to a system $\tilde{\Sigma}$ which is passive with a positive definite and proper storage function. In view of Theorem 3.2, the result will be proven if we show that $\tilde{\Sigma}$ is zero-state detectable. To this end, suppose $(\zeta(t), x(t))$ is any unforced trajectory of $\tilde{\Sigma}$ yielding $y(t) = 0$ for all $t \geq 0$. Since $\zeta(t)$ satisfies (5.2), which is globally asymptotically stable by assumption, $\zeta(t) \to 0$ as $t \to \infty$. On the other hand, $x(t)$ satisfies an equation of the form

$$\dot{x} = f(x) + g(x)\alpha(\zeta(t), x)$$

where $\alpha(x, \zeta)$ is a feedback function which has the form

$$\alpha^{\mathrm{T}}(\zeta, x) = -L_{f_1(\zeta, h(x))}U(\zeta)$$

with U a Lyapunov function for (5.2). Note that, since $\zeta = 0$ is a minimum for U, $\alpha(\zeta, x)$ vanishes at $\zeta = 0$. As in the proof of Proposition 3.4, let γ° denote the ω-limit set of $(\zeta(t), x(t))$. Since $\zeta(t) \to 0$ as $t \to \infty$, any point of γ° is a pair of the form $(0, x)$. Let $(0, \bar{x})$ be one of these points and let $(0, \bar{x}(t))$ denote the corresponding trajectory. Clearly, $\bar{x}(t)$ satisfies

$$\dot{\bar{x}}(t) = f(\bar{x}(t)), \qquad L_g V(\bar{x}(t)) = 0$$

and this together with the fact that $V(\bar{x}(t))$ is constant, yields, as in the proof of Proposition 3.4, $\bar{x}(t) \in S$. Thus, $\tilde{\Sigma}$ is zero-state detectable. △

As an immediate application of this result we obtain, the following.

Corollary 5.2: Consider a system Σ described by

$$\dot{\zeta} = f_0(\zeta) + f_1(\zeta, y) y$$
$$\dot{y} = f(\zeta, y) + g(\zeta, y) u.$$

Suppose $g(\zeta, y)$ is invertible for all ζ, y. Suppose

$$\dot{\zeta} = f_0(\zeta)$$

is globally asymptotically stable. Then Σ is globally asymptotically stabilizable by smooth state feedback.

Proof: The feedback law

$$u = [g(\zeta, y)]^{-1}[-f(\zeta, y) + v]$$

changes Σ into a system satisfying the assumptions of Theorem 5.1. In fact, the driving system thus obtained has the form

$$\dot{y} = v$$

and is indeed passive with positive definite storage function $V(y) = y^{\mathrm{T}} y$ and $S = \{0\}$. △

The system considered in this statement is just a globally minimum phase system with relative degree $\{1, \cdots, 1\}$ represented in its global normal form. Thus, Corollary 5.2 coincides with [3, Theorem 2.1]. One of our next applications consists in showing that the minimum phase assumption of [3, Theorem 2.1] can in fact be weakened, in the sense that also *weakly* minimum phase systems may be globally asymptotically stabilized by smooth feedback. We will prove this result after having shown how Theorem 5.1 specializes in case the driving system has a globally defined normal form.

From Theorem 4.7 we known that a system having a global normal form is feedback equivalent to a passive system if and only if it is globally weakly minimum phase. Thus, in the light of Theorem 5.1 if the driving system has a global normal form it is convenient, for the purpose of asymptotic feedback stabilization of the full system (5.1), to weaken the passivity assumption by only requiring the weak minimum phase property. Proceeding in this way, one may obtain an alternative version of Theorem 5.1, with a different set of assumptions about the driving system. Of course, in addition to the existence of a global normal form and the property of being minimum phase, one should consider also an assumption which replaces the condition $S = \{0\}$.

To this end, recall that if a system

$$\dot{z} = f^*(z) + p(z, y) y \tag{5.3a}$$
$$\dot{y} = b(z, y) + a(z, y) v \tag{5.3b}$$

is globally weakly minimum phase, there exists a C^r, $r \geq 1$, function $W^*(z)$, defined for all z with $W^*(0) = 0$, which is positive definite and proper, and such that $L_{f^*} W(z) \leq 0$ for all z. Set

$$g^*(z) = p(z, 0) \tag{5.4a}$$

and define

$$\mathfrak{D}^* = \text{span}\,\{\text{ad}_{f^*}^k g_i^*, 0 \leq k \leq n - m - 1, 1 \leq i \leq m\} \tag{5.4b}$$

$$S^* = \{z \in Z^* : L_{f^*}^m L_\tau W^*(z) = 0, \text{ for all } \tau \in \mathfrak{D}^*, \text{ all } 0 \leq m < r\}. \tag{5.4c}$$

In the following statement, which expresses the form to which Theorem 5.1 reduces in case *the driving system has a globally defined normal form*, we show that the condition $S^* = \{0\}$ is, in fact, the condition needed, in addition to the global asymptotic stability of (5.2) and the globally weakly minimum phase property of the driving system, to ensure global asymptotic stabilizability.

Theorem 5.3: Consider a system Σ described by

$$\dot{\zeta} = f_0(\zeta) + f_1(\zeta, y) y \tag{5.5}$$
$$\dot{z} = f^*(z) + p(z, y) y \tag{5.6a}$$
$$\dot{y} = b(z, y) + a(z, y) u. \tag{5.6b}$$

Suppose the unforced dynamics of the driven system (5.5) is globally asymptotically stable and suppose the driving system (5.6) has relative degree $\{1, \cdots, 1\}$ at each point and is globally weakly minimum phase. Suppose $S^* = \{0\}$ (where S^* is defined as in (5.4)). Then Σ is globally asymptotically stabilizable by smooth state feedback.

Proof: The proof is similar to that of Theorem 5.1. We know from Section IV that the feedback law

$$u = a^{-1}(z, y)[-b(z, y) - L_{f_1(\zeta, y)} U(\zeta) - L_{p(z, y)} W^*(z) + v]$$

where $U(\zeta)$ is a Lyapunov function for the unforced dynamics of (5.5) and $W^*(z)$ a Lyapunov function for the zero dynamics of (5.6), changes Σ into a system $\tilde{\Sigma}$

$$\dot{\zeta} = f_0(\zeta) + f_1(\zeta, y) y$$
$$\dot{z} = f^*(z) + p(z, y) y$$
$$\dot{y} = -L_{f_1(\zeta, y)} U(\zeta) - L_{p(z, y)} W^*(z) + v$$

which is passive with a positive definite and proper storage function. So, we only need to show that $\tilde{\Sigma}$ is zero-state detectable. Let $(\zeta(t), z(t), y(t))$ be any unforced trajectory yielding $y(t) = 0$ and let γ° denote its ω-limit set. As in the proof of Theorem 5.1, it is immediate to see that any initial condition in γ° produces a trajectory which has the form $(0, \bar{z}(t), 0)$, with $\bar{z}(t)$ satisfying

$$\dot{\bar{z}}(t) = f^*(\bar{z}(t))$$
$$0 = L_{p(\bar{z}, 0)} W^*(\bar{z}(t)) = L_{g^*} W^*(\bar{z}(t)).$$

Thus, since $W^*(\bar{z}(t)))$ is constant, we can deduce that $\bar{z}(t) \in S^*$ and conclude that $\tilde{\Sigma}$ is zero-state detectable. △

Taking the driven system to be trivial in Theorem 5.3, we obtain the following corollary, an extension of [3, Theorem 2.1], which describes conditions under which a *globally weakly minimum phase* system can be globally asymptotically stabilized by smooth feedback.

Corollary 5.4: Suppose the system Σ described by

$$\dot{z} = f^*(z) + p(z, y)y$$
$$\dot{y} = b(z, y) + a(z, y)u$$

has relative degree $\{1, \cdots, 1\}$ at each point and is globally weakly minimum phase. Suppose $S^* = \{0\}$. Then Σ is globally asymptotically stabilizable by smooth state feedback.

Corollary 5.4 deals with the case in which the *driven* system is trivial. Another situation of interest is the one in which the *driving* system is *linear*, a special case of which was analyzed by Byrnes–Isidori in the original proof [3] of Corollary 5.2. The analysis of [3] was generalized, in a rather ingenious way, by Kokotovic and Sussmann, who proved that feedback stabilization is possible if the linear driving system can be rendered positive real by feedback [17]. As we pointed out in Remark 4.8, the characterization we obtained for feedback equivalence of a nonlinear system to a passive system proves, in the case of linear systems, the equivalence (also demonstrated in [24]) of the following conditions:

i) CB is nonsingular and the system is weakly minimum phase;

ii) the system is feedback equivalent to a passive system, with positive definite storage function $V(x) = x^{\mathrm{T}}Qx$;

iii) the system is feedback equivalent to a positive real system.

Accordingly one deduces, as a corollary of Theorem 5.3, the following result originally proven in [17] and [24].

Corollary 5.5: Consider a system Σ described by

$$\dot{\zeta} = f_0(\zeta) + f_1(\zeta, y)y$$
$$\dot{x} = Ax + Bu$$
$$y = Cx.$$

Suppose the unforced dynamics of the driven system is globally asymptotically stable. Suppose (A, B) is controllable and suppose the driving system $\{A, B, C\}$ satisfies either one of the three equivalent conditions i), ii), or iii). Then Σ is globally asymptotically stabilizable by smooth state feedback.

Proof: According to Theorem 5.3 we need only to check the condition $S^* = \{0\}$, but this is a straightforward consequence of the controllability assumption and of the definition of S^*. △

Remark 5.6: Since the original submission of this paper, whose results were announced also in [5], we became aware of the work by Ortega on stabilizability of cascade connected nonlinear systems [19]. Specifically, Ortega considers the interconnection (5.1) with a *strictly passive* driving system and proves that if (5.2) is globally asymptotically stable, feedback stabilization of (5.1) is possible. Strict passivity implies (hypotheses H2 and H3 of [19]) the existence of a C^2 function $V(x)$ such that $h(x) = L_g V(x)$ and $L_f V(x)$ is negative definite. Thus, in particular, $L_f V(x) = 0$ only at $x = 0$ and the set S defined in Section III necessarily satisfies

$$S = \{0\}.$$

Therefore the stabilization result of [19] can be deduced from Theorem 5.1. △

VI. Conclusions

In this paper, we have investigated the conditions under which a nonlinear system can be rendered passive via smooth state feedback. As in the case of linear systems, it turns out that this is possible if and only if the system in question has relative degree 1 and is weakly minimum phase. Passive systems which are "detectable" can be globally asymptotically stabilized by pure gain output feedback. Moreover, the detectability conditions needed to this purpose can be given a form which involves repeated Lie brackets of vector fields characterizing the input-state differential equation, reminiscent of the well-known rank criteria used for accessibility and controllability. As a consequence, "controllable" weakly minimum phase nonlinear systems having relative degree 1 can be globally asymptotically stabilized by smooth state feedback. This result incorporates and extends a number of stabilization schemes recently proposed in the literature for global asymptotic stabilization of certain classes of nonlinear systems.

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