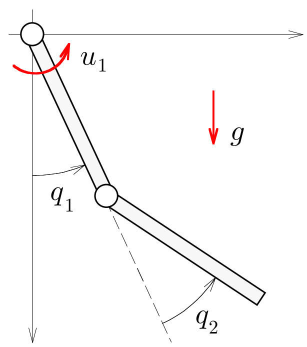
The Pendubot

Leonardo Lanari and Giuseppe Oriolo

The Pendubot is a planar robot moving in the vertical plane; it has two rotational joints and a single actuator at the first joint (shoulder). Since it has more degrees of freedom (two) than control inputs (one), it is an instance of *underactuated* system. The figure shows a schematic diagram of the Pendubot.



1 Mathematical model

The generalized coordinates vector is $q = (q_1, q_2)$, respectively the first joint angle with respect to the downward vertical axis and the second joint angle with respect to the first link axis (*relative* joint coordinates). For the i-th link, denote by m_i, ℓ_i, d_i and I_{izz} respectively its mass, its length, the distance from the $i-th$ joint axis to its center of mass, and the link inertia moment around the z axis passing through the center of mass. Also, denote by u_1 the torque input at the shoulder joint. Following the Euler-Lagrange approach, the nonlinear dynamic equations of the Pendubot are obtained in the classical form

$$M(q)\ddot{q} + F\dot{q} + c(q,\dot{q}) + e(q) = u \tag{1}$$

where $u = (u_1, 0)$, while the inertia matrix $M(q)$, the viscous friction matrix F, the vector $c(q,\dot{q})$ of Coriolis and centrifugal forces and the vector $e(q)$ of gravitational forces and are given as

$$M(q) = \begin{pmatrix} a_1 + 2a_2c_2 & a_3 + a_2c_2 \\ a_3 + a_2c_2 & a_3 \end{pmatrix} \quad F = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} \quad e(q) = \begin{pmatrix} a_4s_1 + a_5s_{12} \\ a_5s_{12} \end{pmatrix} \quad c(q,\dot{q}) = \begin{pmatrix} a_2s_2\,\dot{q}_2(\dot{q}_2 + 2\dot{q}_1) \\ a_2s_2\,\dot{q}_1^2 \end{pmatrix}$$

In these expressions, we have set $s_i = \sin q_i$, $c_i = \cos q_i$, $s_{ij} = \sin(q_i + q_j)$, and

$$\begin{aligned} a_1 &= I_{1zz} + m_1d_1^2 + I_{2zz} + m_2(\ell_1^2 + d_2^2) \\ a_2 &= m_2\,\ell_1d_2 \\ a_3 &= I_{2zz} + m_2d_2^2 \\ a_4 &= g\,(m_1d_1 + m_2\ell_1) \\ a_5 &= gm_2d_2 \end{aligned}$$

where g is the gravity acceleration. All these coefficients, as well as f_1 and f_2, are positive.

See [1] for a general treatment of robot dynamics and a detailed derivation of the dynamic equations of a planar robot with two rotational joints; the above equations are simply obtained by setting $u_2 = 0$ (no actuator at the elbow joint) in those.

2 Control properties

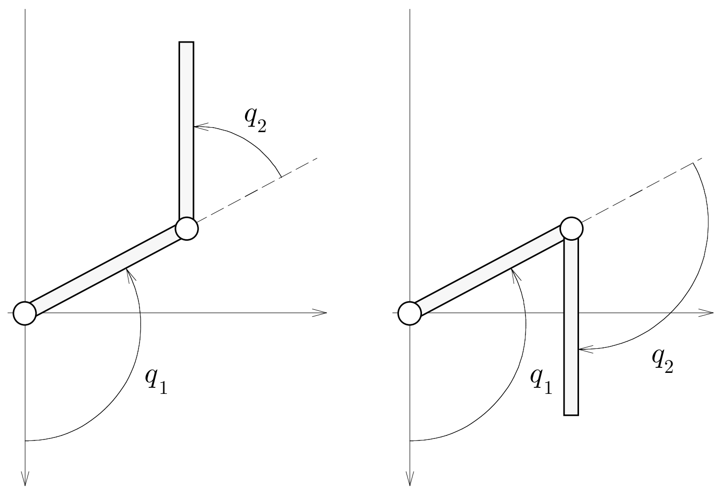
We recall now some basic properties of the Pendubot from a control viewpoint.

2.1 Equilibrium configurations

The forced equilibrium configurations q_e of the Pendubot are identified by setting $\dot{q} = \ddot{q} = 0$ in the dynamic model and solving $e(q_e) = u_e$, with $u_e = (u_{1e}, 0)$ the constant equilibrium torque. We obtain

$$q_e = \begin{pmatrix} q_{1e} \\ q_{2e} \end{pmatrix} = \begin{pmatrix} q_{1e} \\ k\pi - q_{1e} \end{pmatrix} \qquad k = 0, 1$$

with $u_{1e} = a_4 \sin q_{1e}$. For an arbitrary value of q_{1e}, there are thus two equilibrium configurations, as shown in the next figure.



Among the infinite equilibrium configurations, four are *unforced* (i.e., $u_e = 0$). They are clearly the *up-up configuration* $q_{uu} = (\pi, 0)$, the *up-down configuration* $q_{ud} = (\pi, \pi)$, the *down-down configuration* $q_{dd} = (0, 0)$ and the *down-up configuration* $q_{du} = (0, \pi)$.

Let us now study the stability properties of the generic equilibrium point $(q_e, \dot{q}_e = 0)$, with equilibrium torque u_e, using the direct method of Lyapunov. To this end, consider the following function

$$V(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + U(q) - U(q_e) + (q_e - q)^T e(q_e)$$

where $U(q)$ is the potential energy associated to the gravitational field (remember that $e(q) = \partial U/\partial q$). Since the first term is clearly positive definite with respect to $\dot{q}$, a sufficient condition for V to be positive definite is that the Hessian of the remaining part, which reduces to $\partial e/\partial q$, is positive definite around q_e. We have

$$H = \frac{\partial e(q)}{\partial q} = \begin{pmatrix} a_4 c_1 + a_5 c_{12} & a_5 c_{12} \\ a_5 c_{12} & a_5 c_{12} \end{pmatrix}$$

so that necessary and sufficient conditions for positive definitess of H (Sylvester's criterion) are

$$a_4 c_1 + a_5 c_{12} > 0 \qquad \text{and} \qquad \det H = a_4 a_5 c_1 c_{12} > 0$$

around q_e. These are verified if and only if $c_1 > 0$ and $c_{12} > 0$. Hence, V is positive definite at $(q_e, \dot{q}_e = 0)$ in the region $\mathcal{Q} = \{q : q_1 \in (-\pi/2, \pi/2), q_1 + q_2 \in (-\pi/2, \pi/2)\}$; it is therefore a valid candidate Lyapunov function for equilibrium configurations that are strictly contained in such region. These are points of the form $(q_{1e}, -q_{1e})$, with $q_{1e} \in (-\pi/2, \pi/2)$ (first link pointing downwards, second link pointing down along the vertical); in particular, the only unforced equilibrium included in this analysis is q_{dd}.

The derivative of V along system trajectories is computed as

$$\dot{V} = \dot{q}^T M(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + e^T(q)\dot{q} - e^T(q_e)\dot{q} = -\dot{q}^T F \dot{q} \leq 0$$

having used the robot dynamic model (1), with $u = u_e = e(q_e)$, and its properties (namely, the skew-symmetry of $\dot{M}(q) - 2S(q,\dot{q})$, where $c(q,\dot{q}) = S(q,\dot{q})\dot{q}$, and the positive definiteness of F). To compute the maximally invariant set $\mathcal{I}$ contained in $\mathcal{P} = \{(q,\dot{q}) : \dot{V} = 0\} = \{(q,\dot{q}) : \dot{q} = 0\}$, write the dynamics in $\mathcal{P}$

$$M(q)\ddot{q} + e(q) - e(q_e) = 0$$

$\mathcal{I}$ is then defined by $e(q) = e(q_e)$, which implies $q = q_e$ in $\mathcal{Q}$. Wrapping up, La Salle's theorem allows to conclude that forced equilibria with first link pointing downwards and second link pointing down along the vertical are asymptotically stable if F is positive definite, a condition also known as *complete damping*. For each equilibrium, the domain of attraction can be also estimated using the same theorem: in particular, it is easy to verify that zero-velocity points contained in $\mathcal{Q}$ belong to the domain of attraction of any equilibrium in $\mathcal{Q}$.

It is also possible to show that equilibrium points outside $\mathcal{Q}$ (i.e., those with first link pointing downwards and second link pointing *up* along the vertical, as well as those with first link pointing *upwards*) are unstable. This may be proven either using instability criteria (e.g. Chetaev's criterion) or the indirect method of Lyapunov, based on the approximate linearization at the equilibrium point (see Section 2.2).

See [2, 3] for general results on Lyapunov stability, including La Salle's theorem and Chetaev's criterion; see also [4].

2.2 Approximate linearization at equilibria

An elementary approach to the development of controllers for the Pendubot is to derive its approximate linearization at equilibrium points. This will also be useful for gaining some insight into the control properties of the system. To obtain the linear approximate system at a given equilibrium $(q_e, \dot{q}_e = 0)$, with equilibrium torque u_e, one can perform first-order Taylor expansion either on the state-space form corresponding to the dynamic equation (1), or directly on eq. (1) itself. This second approach leads to the following linear equations

$$M(q_e)\ddot{q} + F\dot{q} + \left.\frac{\partial e(q)}{\partial q}\right|_{q=q_e}(q - q_e) = u - u_e \qquad \text{with} \qquad \left.\frac{\partial e(q)}{\partial q}\right|_{q=q_e} = \begin{pmatrix} a_4 c_1 + a_5 c_{12} & a_5 c_{12} \\ a_5 c_{12} & a_5 c_{12} \end{pmatrix}_{q=q_e}$$

Defining the state and input vectors as

$$\tilde{x} = \begin{pmatrix} \tilde{q} \\ \dot{\tilde{q}} \end{pmatrix} = \begin{pmatrix} q - q_e \\ \dot{q} \end{pmatrix} \qquad \tilde{u}_1 = u_1 - u_{1e}$$

and letting $M_e = M(q_e)$, $H_e = \left.\frac{\partial e(q)}{\partial q}\right|_{q=q_e}$, the above second-order linear dynamics is rewritten as

$$M_e\ddot{\tilde{q}} + F\dot{\tilde{q}} + H_e\tilde{q} = \begin{pmatrix} \tilde{u}_1 \\ 0 \end{pmatrix} \tag{2}$$

The linear approximation at $(q_e, \dot{q}_e = 0)$ in state-space form is then

$$\dot{\tilde{x}} = A\tilde{x} + B\tilde{u}_1 = \begin{pmatrix} 0 & I \\ -M_e^{-1}H_e & -M_e^{-1}F \end{pmatrix}\tilde{x} + \begin{pmatrix} 0 \\ M_e^{-1}n \end{pmatrix}\tilde{u}_1 \tag{3}$$

with $n = (1\ \ 0)^T$.

The above linear approximation (either in second-order or in state-space form) can be used to analyze the stability of the original nonlinear system (1) by means of the indirect method of Lyapunov. In particular, we can take advantage of special results for linear mechanical systems whose dynamics have the form (2). If the system is completely damped, asymptotic stability is guaranteed if H_e is symmetric and positive definite. Note that this is a slightly less stringent sufficient condition than the one obtained by the direct method of Lyapunov in Sect. 2.1, which required H to be positive definite *around* q_e. In practice, however, the conclusion is exactly the same, because the only equilibrium configuration for which H_e is positive definite are those contained in $\mathcal{Q}$.

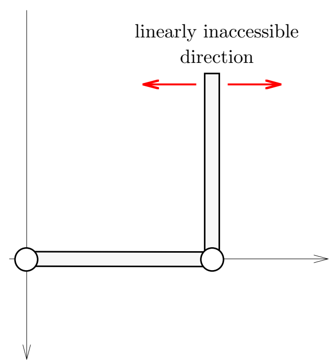
More interesting is the instability analysis. This relies on a result known as Kelvin-Tait-Chetaev theorem, which states that a sufficient condition for instability of the completely damped dynamics (2) is *statical instability*, i.e., the fact that H_e has at least one negative real eigenvalue. One may easily check that this is true for *all* equilibrium configurations outside $\mathcal{Q}$, with the exception of $(q_{1e} = \pm\pi/2, q_{2e} = -q_{1e})$, where a typical *critical* case is met.

If $F = 0$ in eq. 2 (*conservative* system), asymptotic stability is out of the question. A necessary and sufficient condition for simple stability is that H_e is symmetric and definite positive. In our case, this means that equilibrium configurations with the first link pointing downwards and the second link pointing down along the vertical (e.g., q_{dd}) are stable, while all the others are unstable.

The controllability matrix of the linear approximation is (we set $F = 0$ for simplicity, but the result is general):

$$P = \begin{pmatrix} B & AB & A^2B & A^3B \end{pmatrix} = M_e^{-1} \begin{pmatrix} 0 & n & 0 & -H_e M_e^{-1} n \\ n & 0 & -H_e M_e^{-1} n & 0 \end{pmatrix}$$

which is singular when the vectors n and $H_e M_e^{-1} n$ are linearly dependent. One may easily verify that this happens if and only if $q_2 = \pm\pi/2$. Only four equilibrium configurations satisfy this condition, i.e., $(q_{1e} = \pm\pi/2, q_{2e} = \pm\pi/2)$ (see an example in the next figure). In any other equilibrium, the linear approximation is controllable; therefore, the nonlinear system can be locally exponentially stabilized at the equilibrium by the linear control law $\tilde{u}_1 = K\tilde{x}$, with an appropriate choice of K. Note that the overall control torque is $u_1 = u_{1e} + K\tilde{x}$.



Physically, the loss of linear controllability at equilibria such as the one shown in the figure is due to the impossibility of generating instantaneous accelerations of the second link in the horizontal direction. Equivalently, a Kalman decomposition would immediately show that the second link approximate dynamics is uncontrollable. It should be noted that even if the linear approximation is not controllable, the original nonlinear system may still be controllable, and hence stabilizable; however, exponential stability cannot be achieved in any case.

Another structural property of interest is observability with respect to a chosen output. In particular, while joint positions are usually provided by encoders, velocity measures are seldom available in actual manipulators. In this case, the output is $y = \tilde{q} = C\tilde{x}$, with $C = (I \ \ 0)$. It is immediate to verify that the observability matrix is full rank, and therefore the system is completely observable from $\tilde{q}$.

See, e.g., [3] for the linear approximation procedure and related controllability results; see also [5] for a similar analysis applied to a simple pendulum. Stability and instability results for linear mechanical systems, including Kelvin-Tait-Chetaev theorem, can be found in [6]. For a more detailed analysis of the connection between linear controllability and stabilizability, the reader may consult [7].

2.3 Partial feedback linearization

Another general approach to the control of nonlinear systems is a technique called *feedback linearization*: basically, under certain conditions, there exists a state-dependent change of coordinates and of inputs such that the transformed closed-loop system is exactly linear. One typical application of this method leads to the celebrated *computed torque* control for fully actuated robots. When the conditions for exact *input-state* linearization are not met, as for underactuated robots, one may try to achieve a less stringent objective, i.e., *input-output* linearization. Depending on the chosen output y, different control schemes are obtained. When y is part of the state vector, feedback linearization is also called *partial*.

2.3.1 Feedback linearization w.r.t. the first joint

Computing $\ddot{q}_2$ from the second equation of (1) and plugging it into the first one we obtain

$$\left(M_{11} - \frac{M_{12}^2}{M_{22}} \right) \ddot{q}_1 + c_1(q,\dot{q}) + e_1(q) - \frac{M_{12}}{M_{22}} \left(c_2(q,\dot{q}) + e_2(q) \right) = u_1 \tag{4}$$

where M_{ij} is the generic element of the inertia matrix $M(q)$.

Assign to the control input u_1 the following structure

$$u_1 = \left(M_{11} - \frac{M_{12}^2}{M_{22}} \right) \alpha_1 + c_1(q, \dot{q}) + e_1(q) - \frac{M_{12}}{M_{22}} \left(c_2(q, \dot{q}) + e_2(q) \right) \tag{5}$$

where α_1 is an auxiliary control input to be defined. Substituting u_1 in eq. (4) gives

$$\left(M_{11} - \frac{M_{12}^2}{M_{22}} \right) \ddot{q}_1 = \left(M_{11} - \frac{M_{12}^2}{M_{22}} \right) \alpha_1$$

Since the inertia matrix M is nonsingular, we have $\left(M_{11} - M_{12}^2/M_{22} \right) = \det M / M_{22} \neq 0$, and thus the closed-loop system becomes

$$\begin{aligned} \ddot{q}_1 &= \alpha_1 \\ \ddot{q}_2 &= -\frac{1}{M_{22}} \left(c_2(q, \dot{q}) + e_2(q) + M_{12}\alpha \right) \end{aligned}$$

The dynamics of q_1 under the linearizing control (5) reduces to a double integrator driven by the auxiliary input α_1. Since a linear dynamics has been achieved for the first joint through an appropriate choice of the first joint torque, this is an example of the so-called *collocated* feedback linearization. There is, however, an internal nonlinear dynamics which describes the evolution of q_2.

In the above feedback linearization procedure, we have exploited the second-order structure of the dynamic model (1). A more general technique is to differentiate the chosen output (in this case, q_1) until the control input appears in a nonsingular way; the linearizing control is then chosen so as to enforce a linear input-output behavior.

2.3.2 Feedback linearization w.r.t. the second joint

Another possibility is to linearize the dynamics of q_2. Proceeding similarly to the previous case, we find that the following choice of the control input u_1

$$u_1 = \left(M_{12} - \frac{M_{11}M_{22}}{M_{12}} \right) \alpha_2 + c_1(q, \dot{q}) + e_1(q) - \frac{M_{11}}{M_{12}} \left(c_2(q, \dot{q}) + e_2(q) \right) \tag{6}$$

leads to the closed-loop system

$$\begin{aligned} \ddot{q}_1 &= -\frac{1}{M_{12}} \left(c_2(q, \dot{q}) + e_2(q) + M_{22}\alpha_2 \right) \\ \ddot{q}_2 &= \alpha_2 \end{aligned}$$

Since the dynamics of q_2 is now linear, this is called *non-collocated* feedback linearization; the residual nonlinear dynamics concerns in this case q_1.

Note however that the linearizing control input (6) is defined if and only if $M_{12} = a_3 + a_2 c_2 \neq 0$ for all values of q_2. This condition, called *strong inertial coupling*, is certainly satisfied if

$$a_3 > a_2 \qquad \text{i.e.} \qquad I_{2zz} > m_2 d_2 (\ell_1 - d_2)$$

See [8] for a detailed treatment of feedback linearization of nonlinear systems and the related conditions. An intuitive introduction to the subject can be found in [5]. A discussion of partial feedback linearization for general underactuated systems is given in [9].

3 Control problems

We now present the fundamental control problems that can be addressed on the Pendubot; each of them corresponds to a particular control experiment of the REAL Lab, which can be performed either on the virtual or on the physical version of the system.

3.1 Balancing control

The objective of the first control problem is to balance the Pendubot around an equilibrium state $(q_e, \dot{q}_e = 0)$. To this end, one may use the linear approximation (3) at q_e to design a stabilizing control law, which will clearly have local validity for the original nonlinear system (i.e., the control law will be effective only for initial conditions contained in a certain *basin of attraction*). Below, we consider the various viable options depending on the sensory equipment. See Section 4 for a detailed description of the available sensors in the REAL Lab Pendubot.

3.1.1 Stabilization via state feedback

If both joint positions and velocities are measured, it is possible to stabilize system (3) via state feedback at any equilibrium point, with the exception of $(q_{1e} = \pm\pi/2, q_{2e} = \pm\pi/2)$ where linear controllability is lost. This is obtaining by letting $\tilde{u}_1 = K\tilde{x}$, where K is a 1×4 matrix such that $A + BK$ is Hurwitz (i.e., all its eigenvalues have negative real part). By partitioning K as $(K_p \;\; K_d)$, with $K_p, K_d : 1 \times 2$, the resulting control torque is

$$u_1 = u_{1e} + \tilde{u}_1 = u_{1e} + K\tilde{x} = u_{1e} + K_p(q - q_e) + K_d\dot{q}$$

which resembles a proportional-derivative (PD) controller with the addition of constant gravity compensation, i.e., the gravitational torque needed to keep the Pendubot at q_e; note, however, that this control law is essentially different, because all the action takes place on the first joint. K_p and K_d must be such that the eigenvalues of

$$A + BK = \begin{pmatrix} 0 & I \\ M_e^{-1}(nK_p - H_e) & M_e^{-1}(nK_d - F) \end{pmatrix}$$

have negative real part. In particular, it is possible to show that a pure PD at the first joint (i.e., $u_1 = u_{1e} + k_p(q_1 - q_{1e}) + k_d\dot{q}_1$) is not sufficient to stabilize the Pendubot at the up-up configuration q_{uu}, even when the system is completely damped.

One way to compute K is to use *Ackermann*'s formula, i.e.,

$$K = -\gamma p^*(A)$$

where γ is the last row of the inverse controllability matrix P^{-1} and $p^*(A)$ is the matrix polynomial obtained by substituting A for λ in the desired characteristic polynomial $p^*(\lambda)$. Possible choices of the closed-loop eigenvalues are given by canonical placements, such as Butterworth or Bessel configurations.

Another possibility is to compute K in such a way that the quadratic cost functional

$$J = \int_0^\infty \left(\tilde{x}^T Q\tilde{x} + \tilde{u}_1^T R\tilde{u}_1\right) dt$$

is minimized. The corresponding control law $\tilde{u}_1 = K\tilde{x}$, called Linear Quadratic Regulator (LQR), achieves a trade-off between transient performance and energy consumption. This strategy may be particularly convenient in the Pendubot case (as in all nonlinear systems for which a controller is designed on the basis of the linear approximation), because by weighting appropriately the state error $\tilde{x}$ in J one may force the system trajectories to stay close to the set-point q_e, and thus inside the basin of attraction of the linear controller.

3.1.2 Stabilization via output feedback

As already mentioned, velocity variables are rarely accessible in actual manipulators. In this case, a feedback stabilization scheme based on the available output must be used. In linear systems, output feedback stabilization can be obtained by two basic approaches: *(i)* eigenvalue assignment based on the separation principle; *(ii)* compensator design using transfer functions.

The first method relies on the use of an asymptotic observer to obtain an estimate of the state vector $\tilde{x}$. For example, such a device can always be built if $y = \tilde{q}$, as the corresponding linearized system is completely observable (see Section 2.2); in particular, a reduced-order observer may be designed which provides only the estimate of $\dot{\tilde{q}}$ ($\tilde{q}$ is directly available from measurements). Once the estimate is available, it can be used in place of the actual state vector in $\tilde{u}_1 = K\tilde{x}$.

To apply the second method, one needs first an expression of the transfer function between the output and the available input u_1. For simplicity, we shall assume $y = \tilde{q}_1$ (so as to fall in the easier SISO case) and $F = 0$

(conservative system). Clearly, any stabilizing controller derived for this case will also represent a solution for the case $y = \tilde{q}$. A simple calculation yields

$$P(s) = \frac{M_{e,22}s^2 + H_{e,22}}{(M_{e,11}s^2 + H_{e,11})(M_{e,22}s^2 + H_{e,22}) - (M_{e,12}s^2 + H_{e,12})^2}$$

where $M_{e,ij}$ and $H_{e,ij}$ are the generic elements of M_e and H_e, respectively.

Note that in all equilibrium configurations with the second link pointing up (i.e., $q_1 + q_2 = \pi$) we have $H_{e,22} = -a_5$; hence, the corresponding linearized system is *non-minimum phase*, since one of its two real zeros is positive. The analysis of the conservative case in Section 2.2 indicates that the system is also unstable. Hence, the synthesis of a stabilizing output controller using classical methods (e.g., frequency domain or root locus techniques) may prove difficult at these configurations. One way to obtain a solution is to perform pole assignment: in particular, a compensator of the form

$$G(s) = \frac{b_3s^3 + b_2s^2 + b_1s + b_0}{s^3 + a_2s^2 + a_1s + a_0}$$

allows an arbitrary placement of the poles of the closed-loop transfer function $W(s) = G(s)P(s)/(1+G(s)P(s))$.

In the above discussion, we have assumed that the linearized system was completely controllable and observable, i.e., that no zero-pole cancellation took place in $P(s)$. At configurations where this is not the case, one should first verify that the 'hidden' dynamics is asymptotically stable.

4 Physical description of the actual Pendubot

References

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