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MAJORITY RULES AND COALITION STABILITY

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Abstract. We consider a class of symmetric games with externalities across coalitions and show that, under certain regularity conditions, restricting the deviating power to majority guarantees the existence of core-stable allocations. We also show that if majorities can extract resources from minorities, stability requires a supermajority rule, whose threshold is increasing in the extraction power.

Keywords: Majority Rule, Supermajority, Externalities, Core.

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1. Introduction

Apart from being a naturally appealing rule to take collective decisions, the simple majority rule has been shown to satisfy a set of desirable axioms on the largest class of preference domains (Dasgupta and Maskin 2008). Moreover, supermajority rules have been shown to guarantee the existence of a "stable" collective decision under mild restrictions on individual and collective preferences (Caplin and Nalebuff 1988).

In this note we discuss a new, and previously unnoticed, property of majority rules that applies when majority and minorities' members can take actions that affect each other's welfare. Such externalities in collective decisions imply that the expected payoffs of a coalition of agents' depend on the behaviour of remaining agents in the system. This problem is well known and extensively studied in the theory of coalitional games with externalities (Bloch 1996, Ray and Vohra 1997,Yi 1997, Maskin 2003, Ray 2007, Hafalir 2007). In particular, *core* allocations fail to exist even when the game possess strong aggregative properties, such as a notion of convexity accounting for externalities (Hafalir 2007), both when the players in complementary coalitions are assumed to stick together (*delta core* or *core with merging expectations*) or when they organize optimally to maximize their own payoffs (*rational expectations core*). In this note we show that in all symmetric games with externalities that satisfy certain regularity conditions, restricting the blocking power to majority coalitions guarantees that the core is nonempty for all expectations on players' behaviour. Moreover, we show that if majorities can extract resources from minorities, then stability requires to limit the deviation power to supermajorities, and that the required size increases with the extraction power.

2. Setup and Notation

2.1. **The Strategic Form Game.** Let $G = (N, (X, u_i)_{i \in N})$ be a game in strategic form, with finite set players $N = \{1, 2, ..., n\}$, strategy set X_i and payoff function $u_i : X_1 \times .. \times X_n \to \mathbb{R}_+$ for each $i \in N$. We assume that G is *symmetric and monotone* in the following sense:

A.1 *(Symmetry)*. $X_i = X$ for all $i \in N$. Moreover, for all $x \in X^n$ and all permutations $p : N \to N$: $u_{p(i)}\left(x_{p(1)}, ..., x_{p(n)}\right) = u_i\left(x_1, ..., x_n\right)$.

A.2 *(Monotone Externalities)*. One of the following two cases must hold:

1. *Positive Externalities* (PE): $u_i(x)$ increasing in $x_{N \setminus \{i\}}$ for all i and all $x \in X^n$;
2. *Negative Externalities* (NE): $u_i(x)$ decreasing in $x_{N \setminus \{i\}}$ for all i and all $x \in X^n$.

2.2. **Coalitions and Coalitional Worth.** A partition $\Pi = (S_1, S_2, .., S_j, .., S_m)$ of N describes the cooperation patterns in the game G. Let s_j denote the cardinality of S_j for all $j = 1, 2, ..m$. Players belonging to the same coalition in Π are assumed to cooperate to achieve their maximal joint payoff. Across coalitions agents set strategies non-cooperatively. Formally, for each partition $\Pi = (S_1, S_2, ..., S_j, ..., S_m)$ we define the game $G(\Pi)$ with player set $\{1, 2, ...j, .., m\}$, each with strategy set $X_j = X_{S_j}$ and payoff function $U_j = \sum_{i \in S_j} u_i$. Note that the way in which we define the game $G(\Pi)$ implies that our symmetry assumption holds both within and across coalitions in $G(\Pi)$ (compare with the weaker assumption used by Yi 1997).

Under suitable assumptions on payoff functions, reaction functions in $G(\Pi)$ are such that all members within a coalition play the same strategy (see Currarini and Marini 2006). When $G(\Pi)$ possesses a unique Nash Equilibrium, equilibrium payoffs unambiguously define coalitional worths in Π. Note that in the present setting, the partition formed by the grand coalition N always generates the maximal aggregate payoff, and is, in this sense, *efficient*.

2.3. **Core Stability.** We will be interested in efficient outcomes of the game G that are stable against objection by subcoalitions of the set N. Because of the presence of externalities, what a coalition expects to obtain by objecting to a proposed allocation crucially depends on what partition it expects to emerge in response to the objection. Let us denote by $\Pi(S)$ such expected partition, for all $S \subset N$. Specific cases include the *gamma* expectation, where players in $N \backslash S$ split up into singletons, the *delta* expectation, where remaining players merge into the coalition $N \backslash S$, and the *rational* expectation, where remaining players are expected to re-organize in the partition of the set $N \backslash S$ that guarantees them the highest aggregate payoff. Given $\Pi(S)$, a characteristic function $v(S)$ is obtained for each coalition S by considering the Nash equilibrium payoff of S in the game $G(\Pi(S))$.

Definition 1 *The core of the characteristic function game* (N, v) *consists of all efficient allocations* $u \in R^n_+$ *such that* $\sum_{i \in S} u_i \geq v(S)$ *for all* $S \subset N$.

The *core* concept allows all possible coalitions to object. In this paper we impose a constraint coming from the cardinality of coalitions, and in particular the case in which only coalitions with at least a given percentage *beta* of all players are allowed to object to proposed efficient allocations. Formally, we will denote by $\mathcal{M}^{\beta}(N)$ the set of all subsets of N that include strictly more than a given percentage β of the players in N. When $\beta = 50\%$ this reduces to the simple majority rule and, for brevity, we will use the notation $\mathcal{M}(N)$; when $\beta > 50\%$ we will use the term *supermajority*.

Definition 2 *The* $\mathcal{M}^{\beta}$*-core of the characteristic function game* (N, v) *consists of all efficient allocations* $u \in R^n_+$ *such that* $\sum_{i \in S} u_i \geq v(S)$ *for all* $S \subset N$ *such that* $s > \beta \cdot n$.

3. Majorities and Core Stability

We start by recording an important property of the class of games considered here, that was proved in Currarini and Marini (2006). We denote by u_S the per capita payoff for members of coalition S, that is, $u_S = U_S/s$.

Let us also denote by $\mathbf{e}_s$ the unit vector of dimension s. For each $S_j \in \Pi$, let us denote as $x_j(x_1, ., x_{j-1}, x_{j+1}, ., x_m)\mathbf{e}_{s_j}$ the efficient strategy profile played by the members of S_j given the strategy profiles $(x_1, ..., x_{j-1}, x_{j+1}, ..., x_m)$ played by the other coalitions in Π. For brevity, we denote by $x_j(x_k)$ the efficient choice of members of S_j as a function of the choice of members of S_k, keeping all other strategies fixed.

Definition 3 The game $G(\Pi)$ satisfies the "contraction property" if for each two coalitions $S_j, S_k \in \Pi$ such that $s_j = s_k$, the function $x_j(x_k)$ is a contraction.

Lemma 1. *Let* G *be a symmetric monotonic game. Let also* G *have either strategic complements or strategic substitutes and the contraction property. For all partition* Π *and coalitions* S, T *in* Π *such that* $t < s$, *the Nash equilibrium* $x(\Pi)$ *of the game* $G(\Pi)$ *satisfies* $u_S(x(\Pi)) < u_T(x(\Pi))$.

Under the *gamma* expectation, Lemma 1 immediately implies that the efficient equal-split allocation in G belongs to the core of (N, v). In fact, any objecting coalition S would face smaller coalitions in the induced partition $\Pi(S)$, whose members are better of than the members of S by Lemma 1, contradicting efficiency of the equal split allocation in the first place. The same argument does not apply, however, to minority coalitions under the *delta*

expectation, who face a larger coalition in the induced partition $\Pi(S)$. The core under the *delta* expectation may in fact be empty even under the assumptions of Lemma 1.
By ruling out the formation of objecting minorities, the simple majority rule generates conditions under which the argument above applies for all types of expectations. In fact, the formation of a majority S is necessarily followed by the formation of smaller coalitions, independently of the particular expectation $\Pi(S)$, and the existence of profitable deviations would, again, contradict the efficiency of the equal split allocation.

Proposition 1. *Let G be a symmetric monotonic game. Let also G have either strategic complements or strategic substitutes and the contraction property. Then, the $\mathcal{M}$-core of the game G is nonempty for all expectations $\Pi(S)$.*

3.1. **Exploitation of Minorities and Supermajority Rules.** As defined above, the worth of a majority S is fully determined by the actions of its members and by the external effects of the actions taken by the other players in N. Majorities have, in this sense, no coercive power over minorities. This makes the analysis of the previous section somewhat unfit to describe political or voting processes and their stability.
If majorities could extract resources from minorities, the argument of Lemma 1 would not, by itself, guarantee non emptiness of the core. The strategic disadvantage of majorities (as established in Lemma 1) would be counteracted by their extraction power which, if too strong, may undermine stability. In what follows we provide a simple sufficient condition under which supermajority rules can restore core-stability under the *delta* expectation even in the case of minorities' exploitation. A straightforward argument shows that this immediately implies core-stability under all possible expectations. This condition essentially strengthens the property established in Lemma 1 by requiring that the strategic disadvantage of majorities - net of any minority exploitation - increases with their relative size. As we show in section 4, this property is satisfied in several well-known games in which coalition formation and cooperation are a relevant issue.
Let us first define

$$\Delta(S) \equiv u_{N\setminus S}(x(\Pi(S)) - u_S(x(\Pi(S)) \tag{3.1}$$

the difference in per capita payoffs between the coalitions $N\setminus S$ and S in the partition $\Pi(S) = \{S, N\setminus S\}$ at the Nash equilibrium associated to $\Pi(S)$. By Lemma 1, $\Delta(S) > 0$ for $S > N\setminus S$.

A3. *The difference $\Delta(S)$ is increasing in s for all $s \geqq \frac{n}{2}$.*

Consider now the characteristic function $v(S)$ obtained from the game G under the *delta* expectation. We can defined a modified function $v^*(S)$ that encompasses the extraction of an arbitrary majority coalition S of a per capita worth of z from each member of the minority coalition $N\setminus S$. The difference between the per-capita payoffs of a majority coalition S and its complement becomes:

$$\Delta^*(S) \equiv u_{N\setminus S}(x(\Pi(S)) - z - u_S(x(\Pi(S)) - z\frac{(n-s)}{s}$$

Proposition 2. *Let Assumptions **A1**, **A2** and **A3** hold. Let z denote the per-capita worth that majorities can extract from minorities. Then, for each $z < \Delta(N\setminus\{i\})\frac{n-1}{n}$, there exists $1 > \beta > \frac{1}{2}$ such that the $\mathcal{M}^\beta$-core is nonempty. Moreover, β is increasing in z.*

Proof. The expression $\Delta^*(S)$ can be written as follows:

$$\Delta^*(S) = \Delta(S) - z(\frac{n}{s})$$

which under **A.3** is increasing in s for all $s \geqq \frac{n}{2}$. This, together with the assumption that $z < \Delta(N \backslash \{i\}) \frac{n-1}{n}$ (a bound on the extraction power) guarantees that there exists some given size $s < n$ after which $\Delta^*(S) > 0$. Lemma 1 and Proposition 1 apply again for all coalitions of size larger than s. □

4. Examples

In this final section we present three well-know examples of games falling in the present framework and for which assumption **A.3**. holds. For these games the size of supermajority required for stability as a function of extraction power can be easily computed.

4.1. **Oligopoly Games.** In an oligopoly game with identical firms and no synergies each merger (or cartel) behaves as a macro-player (i.e. as a single firm). Therefore, in the partition $\Pi(S) = \{S, T\}$ equilibrium profits are such that $U_{N \backslash S}(x(\Pi(S)) = U_S(x(\Pi(S))$. This implies that $\Delta(S)$ is strictly increasing in s for $s \geq n/2$ and, therefore, Proposition 2 applies. The level of required supermajority increases monotonically with the intensity of minority exploitation. For instance, under linear Cournot oligopoly and normalized demand and cost such that $(a-c)^2 = 1$, for $n = 10$ if the extraction power is $z = 0.0055$ the required supermajority is $s^* = 6$, while if $z = 0.088$, $s^* = 9$. Similar results can be obtained in all games in which, as in Cournot, coalitional worths in $G(\Pi)$ are independent on coalitional sizes.

4.2. **Public Good Games.** Ray & Vohra 1997 consider a game of public goods contribution in which each agent $i \in N$ contributes x_i and receives a payoff $U_i(x) = \sum_{j \in N} x_j - cx_i^2$. The worth of coalition S is $U_S(x) = s \sum_{j \in N} x_j - \sum_{i \in S} cx_i^2$. Computing the equilibrium profile $x(\Pi(S))$ we obtain:

$$\Delta(S) = \frac{2s^2 + (n-s)^2}{4c} - \frac{s^2 + 2(n-s)^2}{4c} = \frac{(2s-n)\,n}{4c},$$

which is positive and monotonically increasing in s for $s > \frac{n}{2}$. Therefore, **A.3** holds and Proposition 2 applies. Note that similar results are obtained in all games for which the function $U_S(x(\Pi(S))$ increases more thamproportionally for $s \in \left[0, \frac{n}{2}\right]$ and less than proportionally for $s \in \left(\frac{n}{2}, n\right]$.

4.3. **Alliances in Contests.** Following a number of recent contributions on alliance formation in contests (see, for instance, Bloch 2011 for a survey), let n players exert effort $e_i \in E_i$ and obtain a payoff $u_i : E^n \to \mathbf{R}_+$ given by

$$u_i(e) = p(e)R - c(e_i)$$

where R is a fixed prize obtained by competing, $c(e_i)$ player i's cost of effort, and

$$\begin{cases} p(e) = e_i \left(\sum_{i \in N} e_i\right)^{-1} \text{ if } \sum_{i \in N} e_i > 0 \\ \qquad \text{and } \frac{1}{|N|} \text{ otherwise} \end{cases}$$

is a contest-success function typical of rent-seeking games (Tullock 1987). The effort of each player affects her probability to access prize (here fixed for simplicity).

When only two coalitions S and T compete for prize, it can be shown that Lemma 1 applies. Moroever, numerical simulations show that, with quadratic costs equal to $c(e_i) = \frac{e_i^2}{2}$

$$\Delta(s) = \frac{\sum_{i \in N \backslash S} \left(e_i \left(\sum_{i \in N} e_i \right)^{-1} - \frac{e_i^2}{2} \right)}{n-s} - \frac{\sum_{i \in S} \left(e_i \left(\sum_{i \in N} e_i \right)^{-1} - \frac{e_i^2}{2} \right)}{s}$$

is increasing in s and the required supermajority increases monotonically with the extraction power. For instance, for $n = 10$,

$$s^* = 6 \text{ for } z = 0.00618$$
$$s^* = 7 \text{ for } z = 0.01589$$
$$s^* = 8 \text{ for } z = 0.03328$$
$$s^* = 9 \text{ for } z = 0.07497.$$

5. Concluding Remarks

We have shown that in symmetric games with no synergies the simple majority rule restricts coalitions' blocking power so to guarantee the existence of core-stable allocations. We have also argued that supermajorities may be required to ensure stability when majorities possess exploitation rights over minorities.

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