Robotics 1

Direct kinematics

Prof. Alessandro De Luca
Kinematics of robot manipulators

- study of ...
  geometric and timing aspects of robot motion, without reference to the causes producing it

- robot seen as ...
  an (open) kinematic chain of rigid bodies interconnected by (revolute or prismatic) joints
Motivations

- functional aspects
  - definition of robot workspace
  - calibration
- operational aspects

Task execution (actuation by motors) \(\leftrightarrow\) Task definition and performance

two different “spaces” related by kinematic (and dynamic) maps

- trajectory planning
- programming
- motion control
Kinematics
formulation and parameterizations

- choice of parameterization $q$
  - unambiguous and minimal characterization of robot configuration
  - $n = \#$ degrees of freedom (dof) = $\#$ robot joints (rotational or translational)

- choice of parameterization $r$
  - compact description of position and/or orientation (pose) variables of interest to the required task
  - usually, $m \leq n$ and $m \geq 6$ (but none of these is strictly necessary)
Open kinematic chains

- $m = 2$
  - pointing in space
  - positioning in the plane

- $m = 3$
  - orientation in space
  - positioning and orientation in the plane

- $m = 5$
  - positioning and pointing in space (like for spot welding)

- $m = 6$
  - positioning and orientation in space
  - positioning of two points in space (e.g., end-effector and elbow)

- $r = (r_1, \ldots, r_m)$

- e.g., it describes the pose of frame $RF_E$

- e.g., the relative angle between a link and the following one
Classification by kinematic type
first 3 dofs only

Cartesian or gantry (PPP)
Cartesian or gantry (PPP)
cylindric (RPP)
cylindric (RPP)
polar or spherical (RRP)
polar or spherical (RRP)
SCARA (RRP)
SCARA (RRP)
articulated or anthropomorphic (RRR)
articulated or anthropomorphic (RRR)

R = 1-dof rotational (revolute) joint
P = 1-dof translational (prismatic) joint
Direct kinematic map

- the structure of the direct kinematics function depends on the chosen \( r \)

\[ r = f_r(q) \]

- methods for computing \( f_r(q) \)
  - geometric/by inspection
  - systematic: assigning frames attached to the robot links and using homogeneous transformation matrices
Direct kinematics of 2R planar robot just using inspection...

\[ q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \quad n = 2 \]

\[ r = \begin{bmatrix} p_x \\ p_y \\ \phi \end{bmatrix} \quad m = 3 \]

\[ p_x = l_1 \cos q_1 + l_2 \cos(q_1 + q_2) \]
\[ p_y = l_1 \sin q_1 + l_2 \sin(q_1 + q_2) \]
\[ \phi = q_1 + q_2 \]

for more general cases, we need a ‘method’!
Numbering links and joints

Joint 1

Link 0 (base)

Joint 2

Link 1

Joint i - 1

Link i - 1

Joint i + 1

Link i

Joint n

Link n

(end effector)

Icon representation of joint types for the manipulator skeleton

- Revolute
- Prismatic
Spatial relation between joint axes

axis of joint $i$

axis of joint $i + 1$

common normal (axis of link $i$)

$a_i = \text{displacement } AB$ between joint axes (always well defined)

$\alpha_i = \text{twist angle}$ between joint axes
— projected on a plane $\pi$ orthogonal to the link axis

$\alpha_i$ with sign (pos/neg)!
Spatial relation between link axes

$\mathbf{link} \; i - 1$

$\mathbf{axis \; of \; joint} \; i$

$\mathbf{link} \; i$

\[ d_i = \text{displacement} \: CD \; (\text{a variable if joint } i \; \text{is prismatic}) \]

\[ \theta_i = \text{angle between link axes} \; (\text{a variable if joint } i \; \text{is revolute}) \]

— projected on a plane $\sigma$ orthogonal to the joint axis
Denavit-Hartenberg (DH) frames

- Joint axis \( i - 1 \)
- Link \( i - 1 \) is moved by joint \( i - 1 \)
- Frame \( RF_i \) is attached to link \( i \)
- Axis of joint \( i \) as common normal to joint axes \( i \) and \( i + 1 \)

Robotics 1
Definition of DH parameters

- unit vector $z_i$ along axis of joint $i + 1$
- unit vector $x_i$ along the common normal to joint $i$ and $i + 1$ axes ($i \to i + 1$)
- $\alpha_i =$ distance $DO_i$, + if oriented as $x_i$, always constant (= ‘length’ of link $i$)
- $d_i =$ distance $O_{i-1}D$, + if oriented as $z_{i-1}$, variable if joint $i$ is PRISMATIC
- $\alpha_i =$ twist angle from $z_{i-1}$ to $z_i$ around $x_i$, + if CCW, always constant
- $\theta_i =$ angle from $x_{i-1}$ to $x_i$ around $z_{i-1}$, + if CCW, variable if joint $i$ is REVOLUTE
DH layout made simple
a popular 3-minute illustration...

Denavit–Hartenberg Reference Frame Layout
Produced by Ethan Tira–Thompson

https://www.youtube.com/watch?v=rA9tm0gTln8

- **note:** the author of this video uses $r$ in place of $a$, and does not add subscripts!
Homogeneous transformation
between successive DH frames (from frame $i - 1$ to frame $i$)

- roto-translation around and along $z_{i-1}$

$$i^{-1}A_i'(q_i) = \begin{bmatrix}
\cos \theta_i & -\sin \theta_i & 0 & 0 \\
\sin \theta_i & \cos \theta_i & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & d_i \\
0 & 0 & 0 & 1 \\
\end{bmatrix}= \begin{bmatrix}
\cos \theta_i & -\sin \theta_i & 0 & 0 \\
\sin \theta_i & \cos \theta_i & 0 & 0 \\
0 & 0 & 1 & d_i \\
0 & 0 & 0 & 1 \\
\end{bmatrix}$$

the product of these two matrices commutes!

rotational joint $\Rightarrow q_i = \theta_i$
prismatic joint $\Rightarrow q_i = d_i$

- roto-translation around and along $x_i$

$$i' A_i = \begin{bmatrix}
1 & 0 & 0 & a_i \\
0 & \cos \alpha_i & -\sin \alpha_i & 0 \\
0 & \sin \alpha_i & \cos \alpha_i & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}$$

always a constant matrix
Denavit-Hartenberg matrix


\[
i^{-1}A_i(q_i) = i^{-1}A_i'(q_i) \quad i' A_i = \begin{bmatrix}
\cos \theta_i & -\cos \alpha_i \sin \theta_i & \sin \alpha_i \sin \theta_i & a_i \cos \theta_i \\
\sin \theta_i & \cos \alpha_i \cos \theta_i & -\sin \alpha_i \cos \theta_i & a_i \sin \theta_i \\
0 & \sin \alpha_i & \cos \alpha_i & d_i \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

compact notation: \( c = \cos, \ s = \sin \)

super-compact notation (if feasible): \( c_i = \cos q_i, \ s_i = \sin q_i \)
Ambiguities in defining DH frames

- **frame 0**: origin and $x_0$ axis are arbitrary
- **frame $n$**: $z_n$ axis is not specified
  - however, $x_n$ must intersect and be chosen orthogonal to $z_{n-1}$
- **positive** direction of $z_{i-1}$ (up/down on axis of joint $i$) is arbitrary
  - choose one, and try to ‘avoid flipping over’ to the next one
- **positive** direction of $x_i$ (back/forth on axis of link $i$) is arbitrary
  - if successive joint axes are incident, we often take $x_i = z_{i-1} \times z_i$
  - when natural, follow the direction ‘from base to tip’
- if $z_{i-1}$ and $z_i$ are **parallel** (common normal not uniquely defined)
  - $O_i$ chosen arbitrarily along $z_i$, still trying to ‘zero out’ parameters
- if $z_{i-1}$ and $z_i$ are **coincident**, normal $x_i$ axis can be chosen at will
  - this case occurs only if the two joints are of different kind (P/R or R/P)
  - again, try using ‘simple values’ (0 or $\pi/2$) for constant angles
Direct kinematics of robot manipulators

description ‘internal’ to the robot using

- \(q = (q_1, ..., q_n)\)
- product of DH matrices

\[ T = T_{w} = \prod_{i=1}^{n} A_i(q_i) \]

alternative representations of the **direct kinematics**

\[ w_T = w_{T_0} A_1(q_1) A_2(q_2) ... A_{n-1}(q_{n-1}) A_n(q_n) \]

\[ r = f_r(q) \]
DH assignment for a SCARA robot

Sankyo SCARA 8438

Sankyo SCARA SR 8447

video
Step 1: joint axes

all parallel (or coincident)

twist angles \( \alpha_i = 0 \) or \( \pi \)

J1 shoulder

J2 elbow

J3 prismatic

\( \equiv \)

J4 revolute

Robotics 1
Step 2: link axes

The vertical ‘heights’ of the link axes are arbitrary (for the time being)

$$a_1, a_2, a_3 = 0$$
Step 3: frames

axes $y_i$ for $i > 0$
are not shown
(nor needed; they form right-handed frames)

Robotics 1
Step 4: DH table of parameters

Note that
- $d_1$ and $d_4$ could be set = 0
- $d_4 < 0$ (opposite to $z_3$)
- also, $q_3 < 0$ in this configuration
Step 5: DH transformation matrices

\[ ^0A_1(q_1) = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & a_1c\theta_1 \\ s\theta_1 & c\theta_1 & 0 & a_1s\theta_1 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ ^1A_2(q_2) = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & a_2c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & a_2s\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ ^2A_3(q_3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ q = (q_1, q_2, q_3, q_4) = (\theta_1, \theta_2, d_3, \theta_4) \]

\[ ^3A_4(q_4) = \begin{bmatrix} c\theta_4 & s\theta_4 & 0 & 0 \\ s\theta_4 & -c\theta_4 & 0 & 0 \\ 0 & 0 & -1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]
Step 6a: direct kinematics

homogeneous matrix $^wT_E$ as product of the $^{i-1}A_i(q_i)$’s

\[
^0A_2(q_1, q_2) = \begin{bmatrix}
c_{12} & -s_{12} & 0 & a_1 c_1 + a_2 c_{12} \\
s_{12} & c_{12} & 0 & a_1 s_1 + a_2 s_{12} \\
0 & 0 & 1 & d_1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
^0A_3(q_1, q_2, q_3) = \begin{bmatrix}
c_{12} & -s_{12} & 0 & a_1 c_1 + a_2 c_{12} \\
s_{12} & c_{12} & 0 & a_1 s_1 + a_2 s_{12} \\
0 & 0 & 1 & d_1 + q_3 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
^wT_E = ^0A_4(q_1, q_2, q_3, q_4) =
\begin{bmatrix}
c_{124} & s_{124} & 0 & a_1 c_1 + a_2 c_{12} \\
s_{124} & -c_{124} & 0 & a_1 s_1 + a_2 s_{12} \\
0 & 0 & -1 & d_1 + q_3 + d_4 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

($^wT_0 = ^4T_E = I$)
Step 6b: direct kinematics

as task vector \( r \in \mathbb{R}^m \)

\[
^0A_4(q_1, q_2, q_3, q_4) = \begin{bmatrix}
c_{124} & s_{124} & 0 & \frac{a_1 c_1 + a_2 c_{12}}{} \\
s_{124} & -c_{124} & 0 & \frac{a_1 s_1 + a_2 s_{12}}{} \\
0 & 0 & -1 & \frac{d_1 + q_3 + d_4}{1} \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

extract \( \alpha_z \)
from \( R(q_1, q_2, q_4) \)

\[
r = \begin{bmatrix}
p_x \\
p_y \\
p_z \\
\alpha_z
\end{bmatrix} = f_r(q) = \begin{bmatrix}
\frac{a_1 c_1 + a_2 c_{12}}{} \\
\frac{a_1 s_1 + a_2 s_{12}}{} \\
d_1 + q_3 + d_4 \\
q_1 + q_2 + q_4
\end{bmatrix} \in \mathbb{R}^4
\]

take \( p \in \mathbb{R}^4 \)
as such from \( p(q_1, q_2, q_3) \)
Stanford manipulator

- 6-dof: 2R-1P-3R (spherical wrist)

Robot with shoulder offset

‘one possible’ DH assignment of frames is shown
determine the associated

- table of DH parameters
- homogeneous transformation matrices
- direct kinematics

write a program for computing the direct kinematics

- numerically (Matlab), given a $\mathbf{q}$
- symbolically (Mathematica, Maple, Symbolic Manipulation Toolbox of Matlab, …)
6-dof: 2R-1P-3R (spherical wrist)

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\alpha_i$</th>
<th>$a_i$</th>
<th>$d_i$</th>
<th>$\theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-\pi/2$</td>
<td>0</td>
<td>$d_1 &gt; 0$</td>
<td>$q_1 = 0$</td>
</tr>
<tr>
<td>2</td>
<td>$\pi/2$</td>
<td>0</td>
<td>$d_2 &gt; 0$</td>
<td>$q_2 = 0$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>$q_3 &gt; 0$</td>
<td>$-\pi/2$</td>
</tr>
<tr>
<td>4</td>
<td>$-\pi/2$</td>
<td>0</td>
<td>0</td>
<td>$q_4 = 0$</td>
</tr>
<tr>
<td>5</td>
<td>$\pi/2$</td>
<td>0</td>
<td>0</td>
<td>$q_5 = -\pi/2$</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>$d_6 &gt; 0$</td>
<td>$q_6 = 0$</td>
</tr>
</tbody>
</table>

Joint variables are in red, while their values in the robot configuration shown are in blue.
KUKA LWR 4+

- 7R (no offsets, spherical shoulder and spherical wrist)

frames and table of DH parameters
homogeneous transformation matrices
direct kinematics
d_1 and d_7 can be set = 0 or not (as needed)

available at DIAG Robotics Lab
KUKA KR5 Sixx R650

- 6R (offsets at shoulder and elbow, spherical wrist)
- Determine frames and table of DH parameters
- Homogeneous transformation matrices
- Direct kinematics

Available at DIAG Robotics Lab
Appendix: Modified DH convention

- a modified version used in J. Craig’s book “Introduction to Robotics”, 1986
  - has $z_i$ axis on joint $i$
  - $a_i$ & $\alpha_i$ = distance & twist angle from $z_{i-1}$ to $z_i$, measured along & about $x_{i-1}$
  - $d_i$ & $\theta_i$ = distance & angle from $x_{i-1}$ to $x_i$, measured along & about $z_i$
  - source of much confusion... if you are not aware of it (or don’t mention it!)
  - convenient with link flexibility: a rigid frame at the base, another at the tip...

\[
\begin{align*}
\begin{bmatrix}
  c\theta_i & -c\alpha_is\theta_i & s\alpha_is\theta_i & a_ic\theta_i \\
  s\theta_i & c\alpha_ic\theta_i & -s\alpha_ic\theta_i & a_is\theta_i \\
  0 & s\alpha_i & c\alpha_i & d_i \\
  0 & 0 & 0 & 1
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
  c\theta_i & -s\theta_i & 0 & a_{i-1} \\
  c\alpha_{i-1}s\theta_i & c\alpha_{i-1}c\theta_i & -s\alpha_{i-1} & -d_is\alpha_{i-1} \\
  s\alpha_{i-1}s\theta_i & s\alpha_{i-1}c\theta_i & c\alpha_{i-1} & d_ic\alpha_{i-1} \\
  0 & 0 & 0 & 1
\end{bmatrix}
\end{align*}
\]

classical (or distal)

modified (or proximal)

planar 2R example

modified DH tends to place frames ‘at the base’ of each link